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On the Foundation of Monopoly in Bilateral Exchange*

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Abstract

We address the problem of monopoly in general equilibrium in a mixed version of a monopolistic two-commodity exchange economy where the monopolist, represented as an atom, is endowed with one commodity and "small traders," represented by an atomless part, are endowed only with the other. We provide a theoretical foundation of the monopoly solution in this bilateral framework through a formalization of an explicit trading process inspired by Pareto (1896) for an exchange economy with a finite number of commodities. Then, we provide a game theoretical foundation of our monopoly solution through a two-stage reformulation of our model. This allows us to prove that the set of the allocations corresponding to a monopoly equilibrium and the set of the allocations corresponding to a subgame perfect equilibrium of the two-stage game coincide. Finally, we give the conditions under which our monopoly solution coincides with that defined by Kats (1974) and those, more restrictive, under which it has the geometric characterization proposed by Schydlowsky and Siamwalla (1966).

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Moreover, we establish the formal relationships between our concept of a monopoly equilibrium and that proposed by Pareto (1896), by redefining the latter in terms of our bilateral exchange setting.
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1 Introduction

To the best of our knowledge, Vilfredo Pareto was the first who gave a formalized treatment of the problem of monopoly for a general pure exchange economy with any finite number of commodities, in the first volume of his *Cours d'économie politique*, published in 1896, pp. 62-68 (henceforth just Pareto (1896)). His monopoly quantity-setting solution rests on the assumption that the monopolist gets no utility from the only commodity he is endowed with, but only cares about the revenue he can obtain by selling it.

Seventy years later, Schydlowsky and Siamwalla (1966) proposed a formulation of the problem of monopoly without any mention to the previous work by Pareto (1896). In the context of a pure exchange economy, they considered a bilateral framework where one commodity is held by one trader behaving as a monopolist while the other is held by a "competitors' community." In contrast to Pareto's analysis, the monopolist desires both commodities. The authors provided a geometrical representation of the monopoly solution as the point of tangency between the monopolist's indifference curve and the offer curve of the competitors' community. They did not mention either the geometrical treatment of the monopoly problem previously given, at a very embryo stage, by Edgeworth (1881).

A few years later, Kats (1974), again without mentioning Pareto (1896), analyzed a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity-setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and with the offer curve of the competitive traders.

In this paper, we provide a theoretical foundation of the monopoly solution by formalizing an explicit trading process inspired to that first sketched by Pareto (1896).

We consider the mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part.

This framework can also be used to represent a finite exchange economy if the atomless part is split into a finite number of types with traders of the same type having the same endowments and preferences.

In our setup, the monopolist acts strategically, making a bid of the commodity he holds in exchange for the other commodity, while the atomless part behaves à la Walras. Given the monopolist's bid, prices adjust to equate the monopolist's bid to the aggregate net demand of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. We define a monopoly equilibrium as a strategy played by the monopolist, corresponding to a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids.

Then, we adapt to our monopoly bilateral exchange context the version of the Shapley window model used by Busetto et al. (2020) and we assume that the atomless part behaves à la Cournot making bids of the commodity it holds. We show that there is no Cournot-Nash equilibrium in the market game generated by the strategic interaction between the monopolist and the atomless part through the Shapley window trading process, thereby confirming an analogous negative result obtained by Okuno et al. (1980, p. 24) for the monopolistic version of their bilateral strategic market game. Moreover, we provide an example exhibiting a bilateral exchange economy which admits a monopoly equilibrium but no Cournot-Nash equilibrium. Our example shows that it is not possible to provide a game theoretical foundation of our monopoly solution in terms of an equivalence between the set of the allocations corresponding to a monopoly equilibrium and the set of the allocations corresponding to a Cournot-Nash equilbrium, in a one-stage setting.

Sadanand (1988, p. 174) started from the negative result about the existence of a Cournot-Nash equilibrium in a one-shot monopolistic bilateral strategic market game obtained by Okuno et al. (1980) and this lead him to introduce a monopoly price-setting solution in a two-stage version of the strategic market game analyzed by those authors.

Following Sadanand (1988), we provide a sequential reformulation of the mixed version of the Shapley window model in terms of a two-stage game with observed actions where the quantity-setting monopolist moves first and the atomless part moves in the second stage, after observing the moves of the monopolist in the first stage. This two-stage reformulation of our model allows us to provide a game theoretical foundation of the quantity-setting monopoly solution: we prove that the set of the allocations corresponding to a monopoly equilibrium and the set of those corresponding to a subgame perfect equilibrium of the two-stage game coincide.

The theoretical framework proposed in this paper to define and analyse monopoly equilibrium in bilateral exchange can be simplified, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, and compared with the standard partial equilibrium analysis of monopoly. Indeed, we show that, if this assumption holds, at an allocation corresponding to a monopoly equilibrium, the utility of the monopolist is maximal in the feasible (with respect to aggregate initial endowments) complement of the offer curve of the atomless part, thereby providing a foundation of the monopoly solution proposed by Kats (1974). Moreover, we show that, if the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, a monopoly equilibrium has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). This result rests on a notion which has a well-known counterpart in partial equilibrium analysis and was also used by Pareto (1896) to formulate his solution to the monopoly problem in exchange economies: the marginal revenue of the monopolist.

Finally, we go deeper into the relationship between our analysis and that proposed by Pareto (1896), by redefining and studying this author's concept of a monopoly equilibrium within our framework of bilateral exchange, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of a monopoly equilibrium. In Section 3, we compare the monopoly equilibrium and the Cournot-Nash equilibrium. In Section 4, we provide a game theoretical foundation of the monopoly solution in a two-stage framework. In Section 5, we discuss the model. In Section 6, we characterize the monopoly equilibrium under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and we discuss the literature related to our monopoly solution. In Section 7, we draw some conclusions and we suggest some further lines of research. The proofs of all the propositions are reported in the appendix.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$.[1] Moreover, we assume that $T \setminus T_0 = \{m\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}^i(m) > 0$, $\mathbf{w}^j(m) = 0$ *and* $\mathbf{w}^i(t) = 0$, $\mathbf{w}^j(t) > 0$, *for each* $t \in T_0$, $i = 1$ *or* 2, $j = 1$ *or* 2, $i \neq j$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R^2_+ \to R$ *is continuous, strongly monotone, and strictly quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $D \times F$ such that $D \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R^2_+ \to R$, *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in R^2_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$*-measurable.*

A price vector is a nonnull vector $p \in R^2_+$. Let $\mathbf{X}^0 : T_0 \times R^2_{++} \to \mathcal{P}(R^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^2_{++}$, $\mathbf{X}^0(t, p) = \operatorname{argmax}\{u(x) : x \in R^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in R^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}(t, p)\, d\mu : \mathbf{x}(\cdot, p)$ is integrable and $\mathbf{x}(t, p) \in$

[1] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

$\mathbf{X}^0(t,p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t,\cdot)$ is nonempty and single-valued, by Assumption 2, it is possible to define the Walrasian demand of traders in the atomless part as the function $\mathbf{x}^0 : T_0 \times R^2_{++} \to R^2_+$ such that $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$ and for each $p \in R^2_{++}$.

We can now state and show the following proposition.

Proposition 1. *Under Assumptions 1, 2, and 3, the function $\mathbf{x}^0(\cdot,p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for each $p \in R^2_{++}$.*

We now provide the definition of a monopoly equilibrium in the bilateral exchange model introduced in the previous section. Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1,2\}$ denote the strategy set of atom m. We denote by $e \in \mathbf{E}(m)$ a strategy of atom m, where e_{ij}, $i,j = 1,2$, represents the amount of commodity i that atom m offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definition.

Definition 1. *Given a strategy $e \in \mathbf{E}(m)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^2 e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^2 e_{ji}\mu(m), \quad (1)$$

$j = 1,2$.

The following proposition shows that market clearing price vectors can be normalized.

Proposition 2. *Under Assumptions 1, 2, and 3, if p is a market clearing price vector, then αp, with $\alpha > 0$, is also a market clearing price vector.*

Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R^2_+ : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

The next proposition shows that the two equations in (1) are not independent.

Proposition 3. *Under Assumptions 1, 2, and 3, given a strategy $e \in \mathbf{E}(m)$, a price vector $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$.*

The next proposition is based on Property (iv) of the aggregate demand of an atomless set of traders established by Debreu (1982, p. 728).

Proposition 4. *Under Assumptions 1, 2, and 3, let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, and which converges to a normalized price vector $\bar{p}$. If $\bar{p}^i = 0$ and $\mathbf{w}^i(m) > 0$, then the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$.*

The following proposition provides a necessary and sufficient condition for the existence of a market clearing price vector. In order to state and prove it, we provide the following preliminary definitions.

Definition 2. *A square matrix C is said to be triangular if $c_{ij} = 0$ whenever $i > j$ or $c_{ij} = 0$ whenever $i < j$.*

Definition 3. *We say that commodities i, j stand in relation Q if $\mathbf{w}^i(t) > 0$, for each $t \in T_0$, and there is a nonnull subset T^i of T_0 such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v_t^i(x^i) + v_t^j(x^j)$, for each $x \in R^2_+$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, for each $t \in T^i$.*[2]

Moreover, we introduce the following assumption.

Assumption 4. *Commodities i, j stand in relation Q.*

Proposition 5. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

We denote by $\pi(\cdot)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(\cdot)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(m, e, p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij} \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\
\mathbf{x}^j(m, e, p) &= \mathbf{w}^j(m), \text{ otherwise},
\end{aligned}$$

$j = 1, 2,$

$$\begin{aligned}
\mathbf{x}^j(t, p) &= \mathbf{x}^{0j}(t, p), \text{ if } p \in \Delta \setminus \partial\Delta, \\
\mathbf{x}^j(t, p) &= \mathbf{w}^j(t), \text{ otherwise},
\end{aligned}$$

[2] In this definition, differentiability is to be understood as continuous differentiability and includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012, p. 58)).

$j = 1, 2$, for each $t \in T_0$.

Given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(m) = \mathbf{x}(m, e, p(e)),$$

$$\mathbf{x}(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$.

The next proposition shows that traders' final holdings constitute an allocation.

Proposition 6. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation.*

We can now provide the definition of a monopoly equilibrium.

Definition 4. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_m(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$.

3 Monopoly equilibrium and Cournot-Nash equilibrium

We now provide the definition of a Cournot-Nash equilibrium in the bilateral exchange model introduced in Section 2, adapting to this framework the version of the Shapley window model used by Busetto et al. (2020) (see also Dickson and Tonin (2021) for a survey of the literature on imperfect competition in bilateral exchange).

A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R^4_+ : \sum_{j=1}^{2} b_{ij} \leq \mathbf{w}^i(t), \; i = 1, 2\}$. We denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1, 2$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we denote by $\bar{\mathbf{B}}$ the matrix such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1, 2$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$.

We need to provide now the following two definitions (see Sahi and Yao (1989)).

Definition 5. *A nonnegative square matrix C is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer k such that $c_{ij}^{(k)} > 0$, where $c_{ij}^{(k)}$ denotes the ij-th entry of the k-th power C^k of C.*

Definition 6. *Given a strategy selection* $\mathbf{b}$*, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \sum_{i=1}^{2} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{2} \bar{\mathbf{b}}_{ji}),\ j = 1, 2. \tag{2}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (2) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise. For each strategy selection $\mathbf{b}$ such that $p(\mathbf{b}) \gg 0$, we assume that the price vector $p(\mathbf{b})$ is normalized.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise},
\end{aligned}$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$. It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model.

Definition 7. *A strategy selection* $\hat{\mathbf{b}}$ *such that* $\bar{\hat{\mathbf{B}}}$ *is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ *and for each* $t \in T$.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

The next proposition provides, for our framework, the same negative result about the existence of a Cournot-Nash equilibrium obtained by Okuno et al. (1980 p. 24) and by Sadanand (1988, p. 174).

Proposition 7. *Under Assumptions 1, 2, 3, and 4, there exists no Cournot-Nash equilbrium.*

Proposition 7 has the relevant consequence that the set of monopoly allocations cannot coincide with the set of Cournot-Nash allocations in a one-stage setting, as confirmed by the following example.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4.* $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{2}x_1 + \sqrt{x_2}$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x_1} + x_2$, *for each* $t \in T_0$. *Then, there is a unique monopoly allocation and no Cournot-Nash allocation.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$ and the unique monopoly allocation is $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$. However, there is no Cournot-Nash allocation, by Proposition 7. ∎

4 Monopoly equilibrium as a subgame perfect equilibrium

Example 1 shows the nonequivalence between the set of monopoly and Cournot-Nash allocations in a one-stage game. The analogous negative result reached by Okuno et al. (1980) lead these authors to conclude that "[...] we are unable to model pure monopoly without a competitive fringe in a useful way in this setup" (see Footnote 1, p. 24). In his pathbreaking analysis of monopoly in mixed exchange economies, Sadanand (1988) already recognized the two stage-flavor of monopoly equilibrium. Taking inspiration from his work, we now introduce a two-stage game where the monopolist moves first and the atomless part moves in the second stage, after observing the moves of the monopolist in the first stage. Therefore, borrowing from Busetto et al. (2008), we provide a sequential reformulation of the mixed version of the Shapley window model introduced in the previous section, in terms of a two-stage game with observed actions, following Fudenberg and Tirole (1991, p. 70).

The game is played in two stages, labelled as 0 and 1. An action correspondence in stage 0 is a correspondence $\mathbf{A}^0 : T \to \mathcal{P}(R^4_+)$ such that $\mathbf{A}^0(m) = \{(a_{ij}) \in R^4_+ : \sum_{j=1}^2 a_{ij} \leq \mathbf{w}^i(m),\ i = 1, 2\}$ and $\mathbf{A}^0(t)$ is the singleton {"do nothing"}, for each $t \in T_0$. An action correspondence in stage 1 is a correspondence $\mathbf{A}^1 : T \to \mathcal{P}(R^4_+)$ such that $\mathbf{A}^1(m)$ is the singleton {"do nothing"} and $\mathbf{A}^1(t) = \{(a_{ij}) \in R^4_+ : \sum_{j=1}^2 a_{ij} \leq \mathbf{w}^i(t),\ i = 1, 2\}$, for each $t \in T_0$. We denote by $a^0(t) \in \mathbf{A}^0(t)$ an action of trader t in stage 0, where $a^0_{ij}(m)$, $i, j = 1, 2$, represents the amount of commodity i that atom m offers in exchange for commodity j. An action selection in stage 0 is a function $\mathbf{a}^0 : T \to R^4_+$, such that $\mathbf{a}^0(t) \in \mathbf{A}^0(t)$, for each $t \in T$. We denote by $a^1(t) \in \mathbf{A}^1(t)$ an action of trader t in stage 1, where $a^1_{ij}(t)$, $i, j = 1, 2$, represents the amount of commodity i that a trader $t \in T_0$ offers in exchange for commodity j. An action selection in stage 1 is a function $\mathbf{a}^1 : T \to R^4_+$, whose restriction on T_0 is integrable, such that $\mathbf{a}^1(t) \in \mathbf{A}^1(t)$, for each $t \in T$. Let S^0 and S^1 denote the sets of all action selections in stage 0 and in stage 1, respectively. Any action selection at the end of a stage determines a history at the beginning of the next stage.

We denote by $\mathbf{h}^0 = \emptyset$ the history at the beginning of stage 0 and by $\mathbf{h}^1$ a history at the beginning of stage 1 where $\mathbf{h}^1 = \mathbf{a}^0$, for some $\mathbf{a}^0 \in S^0$. Let H^0 and H^1 denote the sets of all stage 0 and stage 1 histories, respectively, where $H^0 = \emptyset$ and $H^1 = S^0$ Let $H^2 = S^0 \times S^1$ denote the set of all terminal histories. Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, we denote by $\bar{\mathbf{A}}$ the matrix such that $\bar{\mathbf{a}}_{ij} = \mathbf{a}^0_{ij}(m) + \int_{T_0} \mathbf{a}^1_{ij}(t)\, d\mu$, $i, j = 1, 2$.

We now provide the following definition (see Sahi and Yao (1989)).

Definition 8. *Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \sum_{i=1}^{2} p^i \bar{\mathbf{a}}_{ij} = p^j (\sum_{i=1}^{2} \bar{\mathbf{a}}_{ji}),\ j = 1, 2. \tag{3}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (3) if and only if $\bar{\mathbf{A}}$ is irreducible. Then, we denote by $p(\mathbf{h}^2)$ a function which associates with each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ the unique, up to a scalar multiple, price vector p satisfying (3), if $\bar{\mathbf{A}}$ is irreducible, and is equal to 0, otherwise. For each final history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ such that $p(\mathbf{h}^2) \gg 0$, we assume that the price vector $p(\mathbf{h}^2)$ is normalized.

Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(m, \mathbf{h}^2(m), p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} \mathbf{a}^0_{ji}(m) + \sum_{i=1}^{2} \mathbf{a}^0_{ij}(m) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(m, \mathbf{h}^2(m), p) &= \mathbf{w}^j(m), \text{ otherwise,} \end{aligned}$$

$j = 1, 2,$

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{h}^2(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{a}^1_{ji}(t) + \sum_{i=1}^{2} \mathbf{a}^1_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a terminal history $\mathbf{h}^2 = (\mathbf{a}^0, \mathbf{a}^1)$ and the function $p(\mathbf{h}^2)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{h}^2(t), p(\mathbf{h}^2)),$$

for each $t \in T$. It is straightforward to show that this assignment is an allocation.

We denote by $s(t)$ a strategy of trader t, where $s(t)$ denotes the sequence of functions $\{s^0(t, \cdot), s^1(t, \cdot)\}$ such that $s^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$ and $s^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$. A strategy profile $\mathbf{s}$ is a map which associates with each $t \in T$ a sequence of functions $\{\mathbf{s}^0, \mathbf{s}^1\}$ such that $\mathbf{s}^0(t, \cdot) : H^0 \to \mathbf{A}^0(t)$, $\mathbf{s}^1(t, \cdot) : H^1 \to \mathbf{A}^1(t)$, $\mathbf{s}^0(\cdot, \mathbf{h}^0) \in S^0$, and $\mathbf{s}^1(\cdot, \mathbf{h}^1) \in S^1$, for each $\mathbf{h}^1 \in H^1$. Given a strategy profile $\mathbf{s}$, the functions $\mathbf{s}^0(\cdot, \mathbf{h}^0)$ and $\mathbf{s}^1(\cdot, \mathbf{h}^1)$, for each $\mathbf{h}^1 \in H^1$, are called strategy selections. We denote by $\mathbf{s} \setminus s(t) = \{\mathbf{s}^0 \setminus s(t, \cdot), \mathbf{s}^1 \setminus s^1(t, \cdot)\}$ the strategy profile obtained from $\mathbf{s}^0$ and $\mathbf{s}^1$ by replacing, respectively, $\mathbf{s}^0(t, \cdot)$ with $s^0(t, \cdot)$ and $\mathbf{s}^1(t, \cdot)$ with $s^1(t, \cdot)$. Finally, we denote by $\mathbf{h}^2(\mathbf{s})$ the function which associates with each strategy profile $\mathbf{s}$ the terminal history which corresponds to the action selections $\{\mathbf{a}^0(\mathbf{s}), \mathbf{a}^1(\mathbf{s})\}$ such that $\mathbf{a}^0(\mathbf{s}) = \mathbf{s}^0(\cdot, \mathbf{h}^0)$

and $\mathbf{a}^1(\mathbf{s}) = \mathbf{s}^1(\cdot, \mathbf{h}^1)$, with $\mathbf{h^1} = \mathbf{s}^0(\cdot, \mathbf{h}^0)$, and by $\bar{\mathbf{A}}(\mathbf{s})$ the corresponding aggregate matrix.

We now proceed to consider the subgame represented by the stage 1 of the game outlined above, given the history $\mathbf{h}^1 \in H^1$. Given a strategy $s(t)$ of trader t and a history $\mathbf{h}^1 \in H^1$, we denote by $s|\mathbf{h}^1(t)$ the action such that $s|\mathbf{h}^1(t) = s^1(t, \mathbf{h}^1)$. Given a strategy profile $\mathbf{s}$ and a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1$ the strategy selection such that $\mathbf{s}|\mathbf{h}^1(t) = \mathbf{s}^1(t, \mathbf{h}^1)$, for each $t \in T$. Given a history $\mathbf{h}^1 \in H^1$, we denote by $\mathbf{s}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t)$ the strategy selection obtained from $\mathbf{s}|\mathbf{h}^1$ by replacing $\mathbf{s}|\mathbf{h}^1(t)$ with $s|\mathbf{h}^1(t)$. Finally, we denote by $\mathbf{h}^2(\mathbf{s}|\mathbf{h}^1)$ the function which associates with each strategy selection $\mathbf{s}|\mathbf{h}^1$ the terminal history which corresponds to the action selections $\{\mathbf{a}^0(\mathbf{s}|\mathbf{h}^1), \mathbf{a}^1(\mathbf{s}|\mathbf{h}^1)\}$ such that $\mathbf{a}^0(\mathbf{s}|\mathbf{h}^1) = \mathbf{h}^1$ and $\mathbf{a}^1(\mathbf{s}|\mathbf{h}^1) = \mathbf{s}|\mathbf{h}^1$, and by $\bar{\mathbf{A}}(\mathbf{s}|\mathbf{h}^1)$ the corresponding aggregate matrix.

We are now able to define the notion of subgame perfect equilibrium for the two-stage game described above.

Definition 9. *A strategy profile* $\mathbf{s}^*$ *such that* $\bar{\mathbf{A}}(\mathbf{s}^*)$ *is irreducible is a subgame perfect equilibrium if*

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))) \geq u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^* \setminus s(t))(t), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(t))))),$$

for each $s(t)$ *and for each* $t \in T$, $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ *is irreducible, for each* $\mathbf{h}^1 \in H^1$ *such that* $\mathbf{h}^1(m) > 0$, *and*

$$\begin{aligned}
&u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))) \\
&\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(t))(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(t))))),
\end{aligned}$$

for each $\mathbf{h}^1 \in H^1$, *for each* $s|\mathbf{h}^1(t)$, *and for each* $t \in T$.

A subgame perfect allocation is an allocation $\mathbf{x}^*$ such that $\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))$, for each $t \in T$, where $\mathbf{s}^*$ is a subgame perfect equilibrium.

The following proposition shows the equivalence between the set of monopoly allocations and the set of subgame perfect allocations for our two-stage game.

Proposition 8. *Under Assumptions 1, 2, 3, and 4, the set of monopoly allocations coincides with the set of subgame perfect allocations.*

Besides the mixed two-stage game framework, our analysis of monopoly equilibrium as a subgame perfect equilibrium differs from the one proposed by Sadanand (1988) in that it considers a quantity setting monopolist whereas Sadanand (1988) deals with an endogenously determined price-setting monopoly. We leave for further research a reformulation of our model in terms of a price-setting monopolist and a comparison between the two monopoly configurations.

As reminded above, Busetto et al (2008) proposed a respecification à la Cournot-Walras of the mixed version of the Shapley window model for an exchange economy with a finite number of commodities. Since they obtained the negative result that the set of the Cournot-Walras equilibrium allocations of this respecification does not coincide with the set of the Cournot-Nash allocations of the mixed version of the original Shapley model in a one-stage setting, they provided a further reformulation of the Shapley model as a two-stage game. They showed that the set of the Cournot-Walras equilibrium allocations coincides with the set of the Markov perfect equilibrium allocations of the two-stage reformulation of the Shapley model.

The monopoly model developed in this paper cannot be considered as a two-commodity monopoly version of the Cournot-Walras model proposed by Busetto et al. (2008): this would require in fact that the atomless part, in the aggregate, held both commodities.

In this regard, it is worth noticing that a model of partial monopoly, where a monopolist shares a market with a competitive fringe, was proposed in a pioneering work by Forchheimer (1908) (see also Reid (1979) for a detailed analysis of this work). A two-commodity monopoly version of the Cournot-Walras framework proposed by Busetto et al. (2008), where one commodity is held by the monopolist and a fringe of the atomless part whereas the other commodity is only held by the atomless part, could be interpreted indeed as a bilateral exchange generalization of the partial monopoly model introduced by Forchhemeir (1908). An analysis of the relationship between these two approaches deserves to be developed in the detail and we leave it for further research.

5 Discussion of the model

The analysis of the monopoly problem in bilateral exchange proposed in the previous sections can be simplified by introducing the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and compared, under this restriction, with the standard partial equilibrium analysis of monopoly.

The following proposition states a necessary and sufficient condition for the atomless part's aggregate demand to be invertible.

Proposition 9. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(m) > 0$*. Then,* *the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible if and only if, for each* $x \in R_{++}$, *there is a unique* $p \in \Delta \setminus \partial\Delta$ *such that* $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$.

Let $p^{0i}(\cdot)$ denote the inverse of the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$. The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, there exists a unique price selection.

Proposition 10. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, then there exists a unique price selection* $\mathring{p}(\cdot)$.

By analogy with partial equilibrium analysis, $\mathring{p}(\cdot)$ can be called the inverse demand function of the monopolist. When the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the monopoly equilibrium can be reformulated as in Definition 3, with respect to monopolist's inverse function $\mathring{p}(\cdot)$.

Turchet (2021) started to investigate the issue of the existence of a monopoly equilibrium under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and that traders belonging to the atomless part have an identical CES utility function. In this paper, we do not provide a proof of the existence of a monopoly equilibrium but we use the framework considered by Turchet (2021) in order to assess the role of the assumptions we have made in Section 2 to guarantee an existence result. In particular, Assumption 1 guarantees the pure monopoly nature of the economy while Assumption 3 is a standard measurability assumption extended to mixed exchange economies by Shitovitz (1973). These assumptions must be maintained to assure the basic economic and mathematical consistency of the monopoly model. On the other hand, the following example shows that Assumptions 2 and 4 cannot be weakened or omitted without affecting the existence of monopoly equilibrium. The example exhibits an exchange economy where traders in the atomless part have continuous, monotone, and quasi concave utility functions that do not satisfy Assumption 4: in this case, a monopoly equilibrium does not exist.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1 and 3.* $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \sqrt{x_1} + \sqrt{x_2}$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \min\{x_1, x_2\}$, *for each* $t \in T_0$. *Then,* $u_m(\cdot)$ *satisfies Assumption 2,* $u_t(\cdot)$ *is continuous, monotone, and quasi concave, for each*

$t \in T_0$, *the function* $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ *is invertible,* $\mathring{p}(e) = (1 - e_{12}, e_{12})$, *and there is no monopoly equilibrium.*

Proof. It is straightforward to verify that $u_m(\cdot)$ satisfies Assumption 2 and that $u_t(\cdot)$ is continuous, monotone, and quasi concave. The function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible as $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu = p_2$. Moreover, it is immediate to verify that $\mathring{p}(e) = (1 - e_{12}, e_{12})$. Suppose that the strategy $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium. Then, we have that $\tilde{e} > 0$ and $\mathbf{x}(m, \tilde{e}, p(\tilde{e})) = (1 - \tilde{e}_{12}, 1 - \tilde{e}_{12})$. Let $e' \in \mathbf{E}(m)$ be a strategy such that $0 < e'_{12} < \tilde{e}_{12}$. Then, we have that

$$\begin{aligned} u_m(\mathbf{x}(m, e', p(e'))) &= \sqrt{1 - e'_{12}} + \sqrt{1 - e'_{12}} \\ &> \sqrt{1 - \tilde{e}_{12}} + \sqrt{1 - \tilde{e}_{12}} = u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e}))), \end{aligned}$$

a contradiction. Hence there is no monopoly equilibrium ■

6 Discussion of the literature

We now show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, our model can provide an economic theoretical foundation of the solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974).

Under this assumption, the monopoly equilibrium can be characterized by means of the notion of offer curve of the atomelss part, defined as the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, and that of the notion of feasible complement of the offer curve of the atomless part, defined as the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.

The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the feasible complement of the atomless part's offer curve is a subset of the set of the monopolist's final holdings.

Proposition 11. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ *is invertible, then the feasible complement of the offer curve of the atomless part, the set* $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, *is a subset of the set* $\{x \in R^2_+ : x = \mathbf{x}(m, e, \mathring{p}(e)) \text{ for some } e \in \mathbf{E}(m)\}$, *the set of the final holdings of the monopolist.*

Kats (1974) considered both the cases of a quantity setting and a price setting monopoly in a pure exchange economy where one trader behaves as a monopolist, calling the game and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and the offer curve of the competitive traders. However, he did not propose any explicit trading process which could lead to the monopoly solution. The following proposition, which follows from Proposition 11, establishes that, at a monopoly allocation, the utility of the monopolist is maximal in the feasible complement of the atomless part's offer curve. This way, it provides an explicit economic theoretical foundation of the monopoly solution proposed by Kats (1974).

Proposition 12. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$*, the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, and* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium, then* $u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})))$ *is maximal in the set* $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$.

In order to provide the characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966), we need to introduce also the following assumption.

Assumption 5. $u_a : R^2_+ \to R$ *is differentiable.*

We show that, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, the monopoly equilibrium introduced in Definition 3 has also the geometric characterization previously proposed by Schydlowsky and Siamwalla (1966): at a strictly positive monopoly allocation, the monopolist's indifference curve is tangent to the atomless part's offer curve.[3]

The following proposition shows that the function $h(\cdot)$, defined on R_{++} and such that

$$p^i x^i + p^j x^j = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

where $p = p^{0i}(x^i)$ and $x^j = h(x^i)$, represents the offer curve of the atomless part in the sense that its graph coincides with the atomless part's offer curve.

[3]This characterization of the monopoly equilibrium has been diffusely reproposed in standard textbooks in microeconomics (see, for instance, Varian (2014, p. 619), among others).

Proposition 13. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, then the graph of the function* $h(\cdot)$*, the set* $\{x \in R^2_+ : x^j = h(x^i)\}$*, coincides with the set* $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$*, the offer curve of the atomless part.*

Borrowing from Pareto (1896), we now introduce in our general framework a notion which has a counterpart in partial equilibrium analysis: The marginal revenue of the monopolist.

In the rest of this section, with a slight abuse of notation, given a price vector $(p^i, p^j) \in \Delta \setminus \partial\Delta$, we denote by p the scalar $p = \frac{p^i}{p^j}$, whenever $\mathbf{w}^i(m) > 0$. Suppose that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, and that the function $p^{0i}(\cdot)$ is differentiable. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and we have that $\frac{d\mathring{p}(e)}{de_{ij}} = \frac{dp^{0i}(e_{ij}\mu(m))}{dx^i}\mu(m)$, at each $e \in \mathbf{E}(m)$ such that E is triangular, by Proposition 9. In this context, the revenue of the monopolist can be defined as $\mathring{p}(e)e_{ij}$ and his marginal revenue as $\frac{d\mathring{p}(e)}{de_{ij}}e_{ij} + \mathring{p}(e)$, for each $e \in \mathbf{E}(m)$ such that E is triangular.

Then, in the next proposition, we can provide a formal foundation of the geometric characterization of the monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966). Indeed, our proposition establishes that, at an interior monopoly solution, the slope of the monopolist's indifference curve and the slope of the atomless part's offer curve are both equal to the opposite of the monopolist's marginal revenue. Therefore, the tangency characterization of a monopoly equilibrium is demonstrated.

Proposition 14. *Under Assumptions 1, 2, 3, 4, and 5, if* $\mathbf{w}^i(m) > 0$*, the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, the function* $p^{0i}(\cdot)$ *is differentiable, and* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium such that* $\tilde{e} < \mathbf{w}^i(m)$*, then*

$$-\frac{\frac{\partial u_a(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i},$$

where $\tilde{\mathbf{x}}$ *is the monopoly allocation corresponding to* $\tilde{e}$*.*

Finally, we provide an example that illustrates the geometric characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966).

Example 1′. *Consider the exchange economy specified in Example 1. Then, at the unique monopoly equilibrium* $\tilde{e} \in \mathbf{E}(m)$*, the slope of the indifference* *curve of the monopolist is equal to the opposite of his marginal revenue, which, in turn, is equal to the slope of the function which represents the offer curve of the atomless part.*

Proof. From Example 1, we have that the unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$, $\mathring{p}(\tilde{e}) = 1$, $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$, and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$. Moreover, we have that $x^2 = h(x^1) = -\frac{\sqrt{x}^1}{2} + 1$ and

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = -\frac{1}{2} = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

Pareto (1986) was the first author who gave a formalized treatment of the problem of monopoly for a general pure exchange economy. To better understand the relationship between the analysis developed in the previous sections and that proposed by Pareto (1896), we reformulate now this author's monopoly solution within our framework of bilateral exchange.

Pareto (1896) assumed that, for the monopolist, the commodity he is endowed with is "neutral," i.e., it is a commodity from which he does not get any utility.[4] To incorporate this assumption in our model, we amend Assumption 2 as follows.

Assumption 2′. $u_m(x) = x^j$, *whenever* $\mathbf{w}^i(m) > 0$, $i \neq j$, *and* $u_t : R^2_+ \to R$ *is continuous, strongly monotone, strictly quasi-concave, for each* $t \in T_0$.

It is straightforward to verify that Assumption 2′ implies that the utility function of the monopolist is continuous, monotone, and quasi-concave. As in the previous section, given a price vector $(p^i, p^j) \in \Delta \setminus \partial\Delta$, we denote by p the scalar $p = \frac{p^i}{p^j}$, whenever $\mathbf{w}^i(m) > 0$. Hereafter, we assume that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, whenever $\mathbf{w}^i(m) > 0$. Therefore, the revenue of the monopolist can be defined again as $\mathring{p}(e)e_{ij}$.

According to Pareto (1896), the goal of the monopolist is to maximize his revenue. Therefore, we can provide the following definition of a Pareto monopoly equilibrium.

Definition 10. *Let* $\mathbf{w}^i(m) > 0$. *A strategy* $\hat{e} \in \mathbf{E}(m)$ *such that* $\hat{E}$ *is triangular is a Pareto monopoly equilibrium, with respect to the price selection*

[4]For a discussion of the properties of neutral commodities, see, for instance, Varian (2014).

$\mathring{p}(\cdot)$, *if*

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$.

A Pareto monopoly allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(m) = \mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e}))$ and $\hat{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\hat{e}))$, for each $t \in T_0$, where $\hat{e}$ is a Pareto monopoly equilibrium.

The following proposition shows that, when Assumption 2 is replaced with Assumption $2'$, a strategy of the monopolist is a Pareto monopoly equilibrium if and only if it is a monopoly equilibrium. Moreover, it shows that, if $p^{0i}(\cdot)$ is differentiable whenever $\mathbf{w}^i(m) > 0$, then at a Pareto monopoly solution the monopolist's marginal revenue must be nonnegative.

Proposition 15. *Under Assumptions 1, $2'$, 3, and 4, let $\mathbf{w}^i(m) > 0$. Then, a strategy $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, if and only if it is a monopoly equilibrium, with respect to the same price selection. Moreover, if the function $p^{0i}(\cdot)$ is differentiable, and $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, then*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

We now provide an example of a Pareto monopoly equilibrium.

Example 3. *Consider the following specification of an exchange economy satisfying Assumptions 1, $2'$, 3, 4. $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = x_2$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x_1} + x_2$, for each $t \in T_0$. Then, there is a unique Pareto monopoly equilibrium $\hat{e} \in \mathbf{E}(m)$ such that*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) > 0.$$

Proof. The unique Pareto monopoly equilibrium is the strategy $\hat{e}$ such that $\hat{e}_{12} = 1$, $\mathring{p}(\hat{e}) = \frac{1}{2}$, $\hat{\mathbf{x}}(m) = (0, \frac{1}{2})$, $\hat{\mathbf{x}}(t) = (1, \frac{1}{2})$, for each $t \in T_0$. Moreover, we have that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) = \frac{1}{4}.$$

■

Comparing the monopoly solution of Example 1 with the Pareto monopoly solution of Example 3, we can observe that the atomless part is better off at the Pareto monopoly solution than at the monopoly solution as

$$u_t(\hat{\mathbf{x}})(t) = \frac{3}{2} > \frac{5}{4} = u_t(\tilde{\mathbf{x}})(t),$$

for each $t \in T_0$. Moreover, Example 3 shows that when the utility function of the monopolist is continuous, monotone, and quasi-concave a monopoly equilibrium may exist whereas Example 2 showed that this is not the case when those weaker assumptions than those imposed by Assumption 2 hold for the atomless part.

7 Conclusion

In this paper, we have provided a general economic and a game theoretical foundation of the quantity-setting monopoly solution in bilateral exchange which, to the best of our knowledged, was a gap in the literature on monopoly in general equilibrium. Then, we have shown that the *ad hoc* monopoly solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974) fit well in suitable specifications of our general model, as well as the *ante litteram* solution proposed by Pareto (1986).

We leave for future research addressing the problem of a price-setting monopolist, in the same bilateral framework as used in this paper. This goal could be pursued by drawing inspiration from another pioneering work by Vilfredo Pareto (see Pareto (1909)) and could lead to a game theoretical foundation of a monopoly solution of this type in a two-stage setup, as suggested by Sadanand (1988).

Kats (1974), in his final remarks (see p. 31), raised the question of the relationship between monopoly equilibrium and cooperative game theory. He formalized a monopolistic market game based on the notion of a monopolistic quasi-core. He mentioned Shitovitz (1973) as the only other work offering a contribution on this issue. Shitovitz (1973), in his Example 1, actually showed that, in the mixed version of a monopolistic two-commodity exchange economy, the set of allocations in the core does not coincide with the set of Walrasian allocations. This example raised the question whether the core solution to monopolistic market games is "advantageous" or "disadvantageous" for the monopolist (see Aumann (1973), Drèze et al. (1977), Greenberg and Shitovitz (1977), among others). The same issue could be analysed using our monopoly equilibrium solution.

8 Appendix

Proof of Proposition 1. Let $p \in R^l_{++}$. Then, the graph of the correspondence $\mathbf{X}(\cdot, p)$, $\{(t, x) : x \in \mathbf{X}(\cdot, p)\}$, is a subset of $\mathcal{T} \bigotimes \mathcal{B}$, by the same argument as that used by Busetto et al. (2011) (see the proof of their Proposition). But then, by the measurable choice theorem in Aumann (1969), there exists a measurable function $\bar{\mathbf{x}}(\cdot, p)$ such that, $\bar{\mathbf{x}}(t, p) \in \mathbf{X}(t, p)$, for each $t \in T_0$, which is also integrable as $\bar{\mathbf{x}}^j(t, p) \leq \frac{\sum_{i=1}^l p^i \mathbf{w}^i(t)}{p^j}$, $j = 1, 2$, for each $t \in T_0$. We must have that $\mathbf{x}^0(\cdot, p) = \bar{\mathbf{x}}(\cdot, p)$ as $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$. Hence, the function $\mathbf{x}^0(\cdot, p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in R^2_{++}$. ■

Proof of Proposition 2. It straightforwardly follows from homogeneity of degree zero of the function $\mathbf{x}^0(t, \cdot)$, for each $t \in T_0$, and from (1). ■

Proof of Proposition 3. Let a strategy $e \in \mathbf{E}(m)$ be given. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$. Let $p \in \Delta \setminus \partial\Delta$ be a price vector. Suppose that p is market clearing for $j = 1$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{01}(t, p)\, d\mu = e_{12}\mu(m).$$

We have that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t, p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t, p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

as $p^1\mathbf{x}^{01}(t, p) + p^2\mathbf{x}^{02}(t, p) = p^2\mathbf{w}^2(t)$, by Assumption 2, for each $t \in T_0$. Then, we have that

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t, p) + e_{12}\mu(m)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Therefore, p is market clearing for $j = 2$. Suppose now that (1) is satisfied for $j = 2$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t, p) + e_{12}\mu(m)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

But then, we have that

$$p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t, p) + p^1 e_{12}\mu(m) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

On the other hand, we know from the previous argument that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Then, we obtain that

$$\int_{T_0} \mathbf{x}^{01}(t,p) = e_{12}\mu(m).$$

Therefore, p is market clearing for $j = 1$. Hence, $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$. ■

Proof of Proposition 4. According to Debreu (1982), we let $|x| = \sum_{i=1}^{2} |x^i|$, for each $x \in R^2_+$, and $d[0, V] = \inf_{x \in V} |x|$, for each $V \subset R^2_+$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$. Suppose, without loss of generality, that $\bar{p}^1 = 0$ and $\mathbf{w}^1(m) > 0$. Then, we have that $\bar{p}^2 = 1$. But then, the sequence $\{d[0, \mathbf{X}^0(t, p^n)]\}$ diverges to $+\infty$ as $\bar{p}^2\mathbf{w}^2(t) > 0$, for each $t \in T_0$, by Lemma 4 in Debreu (1982, p. 721). Therefore, the sequence $\{d[0, \int_{T_0} \mathbf{X}^0(t, p^n)\, d\mu]\}$ diverges to $+\infty$, by the argument used in the proof of Property (iv) in Debreu (1982, p. 728). This implies that the sequence $\sum_{i=1}^{2}\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ as $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Suppose that the sequence $\{\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu > \int_{T_0} \mathbf{w}^2(t)\, d\mu$, for each $n \geq n_0$. But we have that $\mathbf{x}^{02}(t,p) \leq \mathbf{w}^2(t)$, for each $t \in T_0$ and for each $p \in \Delta \setminus \partial\Delta$, a contradiction. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Hence, the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ whenever $\bar{p}^i = 0$ and $\mathbf{w}^i(m) > 0$. ■

Proof of Proposition 5. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and let $e \in \mathbf{E}(m)$ be a strategy. Suppose that there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ and that the matrix E is not triangular. Then, it must be that $e_{12} = 0$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p)\, d\mu = 0$ as $\mu(T^2) > 0$, by (1). Consider a trader $\tau \in T^2$. We have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} = +\infty$ as 2 and 1 stand in the relation Q, by Assumption 4, and $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} \leq \lambda \hat{p}^1$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\mathbf{x}^{02}(\tau,p) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $p\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^2} = \lambda p^2$, by the necessary conditions of the Kuhn-Tucker theorem. But then, it must be that

$\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, the matrix E must be triangular. Suppose now that E is triangular. Then, it must be that $e_{12} > 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}^1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu > e_{12}\mu(m)$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu > e_{12}\mu(m)$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e_{12}\mu(m)$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu = e_{12}\mu(m)$. Then, q is market clearing as it is market clearing for $j = 1$, by Proposition 3. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu \neq e_{12}\mu(m)$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e_{12}\mu(m)$ as $\mathbf{x}^{01}(t, q) \leq \frac{q^2\mathbf{w}^2(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t, q)\, d\mu < e_{12}\mu(m) < \int_{T_0} \mathbf{x}^{01}(t, p^{n_0})\, d\mu$. Let $O \subset \Delta \setminus \partial\Delta$ be a compact and convex set which contains p^{n_0} and q. Then, the correspondence $\int_{T_0} \mathbf{X}^0(t, \cdot)\, d\mu$ is upper hemicontinuous on O, by the argument used in the proof of Property (ii) in Debreu (1982, p. 728). But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu\}$ is continuous on O as $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Therefore, there is a price vector $\check{p} \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t, \check{p})\, d\mu = e_{12}\mu(m)$, by the intermediate value theorem. Then, $\check{p}$ is market clearing as it is market clearing for $j = 1$, by Proposition 3. Hence, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular. ■

Proof of Proposition 6. Let a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$ be given. Suppose that E is not triangular. Then, we have that $\mathbf{x}(m) = \mathbf{x}(m, e, p(e)) = \mathbf{w}(m)$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e)) = \mathbf{w}(t)$, for each $t \in T_0$ as $p(e) = 0$. Suppose that E is triangular. Then, we have that

$$\int_T \mathbf{x}^j(t)\, d\mu = (\mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j})\mu(m) + \int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu$$
$$= \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1, 2$, as $p(e)$ is market clearing. Hence, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(m)$, the assignment $\mathbf{x}(m) = \mathbf{x}(m, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation. ■

Proof of Proposition 7. Suppose, without loss of generality, that $\mathbf{w}^1(m) >$

0 and that $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. Then, we have that $\mathbf{x}(m, \hat{\mathbf{b}}(m), p(\hat{\mathbf{b}})) = (\mathbf{w}^1(m) - \hat{\mathbf{b}}_{12}(m), \bar{\hat{\mathbf{b}}}_{21})$. Let $b'(m)$ be a strategy such that $0 < b'_{12}(m) < \hat{\mathbf{b}}_{12}(m)$. Then, we have that

$$u_m(\mathbf{x}(m, \hat{\mathbf{b}} \setminus b'(m), p(\hat{\mathbf{b}} \setminus b'(m)))) > u_m(\mathbf{x}(m, \hat{\mathbf{b}}(m), p(\hat{\mathbf{b}}))),$$

as $\mathbf{x}(m, \hat{\mathbf{b}} \setminus b'(m), p(\hat{\mathbf{b}} \setminus b'(m))) = (\mathbf{w}^i(m) - b'_{12}(m)), \bar{\hat{\mathbf{b}}}_{21})$ and $u_m(\cdot)$ is strongly monotone, by Assumption 2, a contradiction. Hence, there is no Cournot-Nash equilibrium. ■

Proof of Proposition 8. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$. Let $\tilde{\mathbf{x}}$ be a monopoly allocation. Then, we have that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$. Consider, first, stage 1 of the game. Let $e \in \mathbf{E}(m)$ be a strategy selection and let $\mathbf{h}^1$ be a history at the beginning of stage 1 of the game such that $\mathbf{h}^1(m) = e$. Suppose that E is triangular. Then, we have that $p(e) \gg 0$ and $p(e)\mathbf{x}^0(t, p(e)) = p(e)\mathbf{w}^2(t)$, for each $t \in T_0$, by Assumption 2. But then, there exist $\lambda^j(t) \geq 0$, $j = 1, 2$, $\sum_{j=1}^2 \lambda^j(t) = 1$, such that

$$\mathbf{x}^0(t, p(e)) = \lambda^j(t) \frac{p^2(e)\mathbf{w}^2(t)}{p^j(e)},$$

$j = 1, 2$, for each $t \in T_0$, by Lemma 5 in Codognato and Ghosal (2000). Let $\boldsymbol{\lambda} : T_0 \to R^2_+$ be a function such that $\boldsymbol{\lambda}^j(t) = \lambda^j(t)$, $j = 1, 2$, for each $t \in T_0$. It is straightforward to show that the function $\mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)$, $i, j = 1, 2$, for each $t \in T_0$, is integrable on T_0. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a strategy selection of the subgame represented by the stage 1 of the game such that $\mathbf{a}^1(\tilde{\mathbf{s}}|\mathbf{h}^1)(m) = \{\text{“do nothing”}\}$ and $\mathbf{a}^1_{ij}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) = \mathbf{w}^i(t)\boldsymbol{\lambda}^j(t)$, $i, j = 1, 2$, for each $t \in T_0$. It is immediate to verify that $(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) \in \mathbf{A}^1(t)$, for each $t \in T$. Consider the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$. We have that

$$\bar{\mathbf{a}}_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1) = \mathbf{a}^0_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1)(m)\mu(m) + \int_{t \in T_0} \mathbf{a}^1_{12}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t)\, d\mu = e_{12}\mu(m) > 0.$$

By the same argument used in the proof of Proposition 5, Assumption 4 implies that $\mathbf{x}^{01}(t, p(e)) > 0$, for each $t \in T^2$. Then, we have that $\boldsymbol{\lambda}^1(t) > 0$, for each $t \in T^2$. But then, we have that

$$\begin{aligned}\bar{\mathbf{a}}_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1) &= \mathbf{a}^0_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(m)\mu(m) + \int_{t \in T_0} \mathbf{a}^1_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t)\, d\mu \\ &= \int_{t \in T_0} \mathbf{w}^2(t)\boldsymbol{\lambda}^1(t)\, d\mu > 0.\end{aligned}$$

Therefore, the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is irreducible. Then, from (1), we obtain that

$$\begin{aligned}\int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu &= \int_{T_0} \boldsymbol{\lambda}^1(t) \frac{p^2(e)\mathbf{w}^2(t)}{p^1(e)}\, d\mu \\ &= \int_{T_0} \mathbf{a}^1_{21}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) \frac{p^2(e)}{p^1(e)}\, d\mu = e_{12}\mu(m).\end{aligned}$$

But then, it must be that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))$ as $p(e)$ satisfies (3) and the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is irreducible. Therefore, it is straightforward to verify that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{x}(m, e, p(e))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. It remains now to show that no trader $t \in T$, in stage 1 of the game, has an advantageous deviation from $\tilde{\mathbf{s}}|\mathbf{h}^1$. This is trivially true for m. Suppose that there exist a trader $\tau \in T_0$ and a strategy $s(\tau)$ such that

$$\begin{aligned}&u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))))) \\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))).\end{aligned}$$

It is straightforward to verify that Definition 8 implies that $p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1)(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))$. Then, we have that

$$\begin{aligned}&u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) \\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) = u_\tau(\mathbf{x}(\tau, p(e))).\end{aligned}$$

It is also immediate to verify that

$$p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))\mathbf{w}(\tau).$$

Then, we have that

$$u_\tau(\mathbf{x}(\tau, p(e))) > u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))),$$

a contradiction. Therefore, it must be that

$$\begin{aligned}&u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) \\ &\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))))),\end{aligned}$$

for each $t \in T_0$.

Suppose that E is not triangular. Then, we have that $p(e) = 0$. Let $\tilde{\mathbf{s}}|\mathbf{h}^1$ denote a strategy selection of the subgame represented by the stage 1 of the game such that $\mathbf{a}^1(\tilde{\mathbf{s}}|\mathbf{h}^1)(m) = \{$"do nothing"$\}$ and $\mathbf{a}^1_{ij}(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) = 0$, $i, j = 1, 2$, for each $t \in T_0$. It is immediate to verify that $(\tilde{\mathbf{s}}|\mathbf{h}^1)(t) \in \mathbf{A}^1(t)$, for each $t \in T$ and that the matrix $\bar{\mathbf{A}}(\tilde{\mathbf{s}}|\mathbf{h}^1)$ is not irreducible. Then, it must be that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)$. Therefore, we have that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\tilde{\mathbf{s}}|\mathbf{h}^1)) = \mathbf{w}^1(m) = \mathbf{x}(m, e, p(e))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p((\tilde{\mathbf{s}}|\mathbf{h}^1))) = \mathbf{w}^2(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. It remains now to show that no trader $t \in T$, in stage 1 of the game, has an advantageous deviation from $\tilde{\mathbf{s}}|\mathbf{h}^1$. This is trivially true for m. Suppose that there exist a trader $\tau \in T_0$ and an strategy $s(\tau)$ such that

$$\begin{aligned}&u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau)))))\\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))).\end{aligned}$$

Then, we have that

$$\begin{aligned}&\mathbf{w}^2(\tau) = u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))\\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)))) = \mathbf{w}^2(\tau),\end{aligned}$$

as $p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)) = 0$, a contradiction. Therefore, we conclude that $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1)$ is irreducible, for each $\mathbf{h}^1 \in H^1$ such that $\mathbf{h}^1(m) > 0$, and

$$\begin{aligned}&u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1))))\\ &\geq u_t(\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))(t), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\mathbf{h}^1(t))))),\end{aligned}$$

for each $\mathbf{h}^1 \in H^1$, for each $s|\mathbf{h}^1(t)$, and for each $t \in T$. Consider now stages 0 and 1 of the game. Let $\tilde{\mathbf{s}}$ be a strategy profile such that $\tilde{\mathbf{s}}(m, \mathbf{h}^0) = \tilde{e}$ and $\tilde{\mathbf{s}}(t, \mathbf{h}^0) = \{$"do nothing"$\}$, for each $t \in T_0$, and $\tilde{\mathbf{s}}(t, \mathbf{h}^1) = (\tilde{\mathbf{s}}|\mathbf{h}^1)(t)$, for each $\mathbf{h}^1 \in H^1$, and for each $t \in T$. Let $\tilde{\mathbf{h}}^1$ be such that $\tilde{\mathbf{h}}^1(m) = \tilde{e}$. We have that $\mathbf{h}^2(\tilde{\mathbf{s}}) = \mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$ as $\mathbf{a}^0(\tilde{\mathbf{s}}) = \tilde{\mathbf{s}}^0(\cdot, \mathbf{h}^0) = \tilde{\mathbf{h}}^1 = \mathbf{a}^0(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$ and $\mathbf{a}^1(\tilde{\mathbf{s}}) = \tilde{\mathbf{s}}^1(\cdot, \tilde{\mathbf{h}}^1) = \tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 = \mathbf{a}^1(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)$. Then, it must be that $p(\tilde{e}) = p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)) = p(\mathbf{h}^2(\tilde{\mathbf{s}}))$. But then, it is straightforward to verify that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}}))) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$$

and

$$\mathbf{x}(t, \mathbf{h}^2(\tilde{\mathbf{s}}(t)), p(\mathbf{h}^2(\tilde{\mathbf{s}}))) = \mathbf{x}(t, p(\tilde{e})),$$

for each $t \in T_0$. Suppose that there exists a strategy $s(m)$ of the monopolist such that

$$u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))))) > u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}})))).$$

Let $e = \tilde{\mathbf{s}}^0 \setminus s(m, \mathbf{h}^0)(m)$. Then, we have that $p(e) = p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m)))$ by the same argument used before. But then, we have that

$$\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m)))) = \mathbf{x}(m, e, p(e)).$$

Therefore, it must be that

$$\begin{aligned} &u_m\mathbf{x}(m, e, p(e)) = u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))(m), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(m))))) \\ &> u_m(\mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}})(m), p(\mathbf{h}^2(\tilde{\mathbf{s}})))) = \mathbf{x}(m, \tilde{e}, p(\tilde{e})), \end{aligned}$$

a contradiction. Suppose that there exist a trader $\tau \in T_0$ and a strategy $s(\tau)$ such that

$$\begin{aligned} &u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))))) \\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}})))). \end{aligned}$$

It is straightforward to verify that Definition 8 implies that $p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))) = p(\mathbf{h}^2(\tilde{\mathbf{s}}))$. Then, we have that

$$\begin{aligned} &u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1 \setminus s|\tilde{\mathbf{h}}^1(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1 \setminus s|\tilde{\mathbf{h}}^1(\tau))))) \\ &= u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}} \setminus s(\tau))))) \\ &> u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}})))) \\ &= u_\tau(\mathbf{x}(\tau, \mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)(\tau), p(\mathbf{h}^2(\tilde{\mathbf{s}}|\tilde{\mathbf{h}}^1)))), \end{aligned}$$

a contradiction. Thus the set of monopoly allocations is a subset of the set of subgame perfect allocations. Let $\mathbf{x}^*$ be a subgame perfect allocation. Then, we have that $\mathbf{x}^* = \mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*)(t), p(\mathbf{h}^2(\mathbf{s}^*)))$, for each $t \in T$, where $\mathbf{s}^*$ is a subgame perfect equilibrium. Let $p(e)$ be a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, the price vector $p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))$ corresponding to the history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$. Let $e \in \mathbf{E}(m)$ be a strategy selection. Suppose that E is triangular. Then, it must that $p(e) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)) \gg 0$ as the matrix $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ is irreducible. Suppose that E is not triangular.

Then, it must be that $p(e) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)) = 0$ as the matrix $\bar{\mathbf{A}}(\mathbf{s}^*|\mathbf{h}^1))$ is not irreducible. It is straightforward to verify that

$$\mathbf{x}(m, e, p(e)) = \mathbf{x}(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))),$$

for each strategy selection $e \in \mathbf{E}(m)$ and for each history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$. It is also straightforward to show that

$$u_t(\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))) > u_t(y),$$

for all $y \in \{x \in R^2_+ : p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))x = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))\mathbf{w}^2(t)\}$, for each $\mathbf{h}^1 \in H^1$ such that $\mathbf{h}^1(m) > 0$ and for each $t \in T_0$, by the same argument used by Codognato and Ghosal (2000) in the proof of their Theorem 2 p. 49. Then, we have that

$$\mathbf{x}(t, p(e)) = \mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))),$$

for each strategy $e \in \mathbf{E}(m)$, for each history $\mathbf{h}^1$ such that $\mathbf{h}^1(m) = e$, and for each $t \in T_0$. Let $e \in \mathbf{E}(m)$ be a strategy selection such that E is triangular and let $\mathbf{h}^1$ be a history such that $\mathbf{h}^1(m) = e$. Then, we have that

$$\begin{aligned}
&\int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu + \mathbf{x}^1(m, e, p(e)) = \int_{T_0} \mathbf{x}^1(t, p(e))\, d\mu + e_{12} \\
&= \int_{T_0} \mathbf{x}^1(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))\, d\mu \\
&+ \mathbf{x}^1(m, \mathbf{h}^2(\tilde{\mathbf{s}}|\mathbf{h}^1)(m), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1))) = \mathbf{w}^1(m)
\end{aligned}$$

as the assignment $\mathbf{x}(t, \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)(t), p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^1)))$, for each $t \in T$, is an allocation. But then, $p(e)$ satisfies (1) by Proposition 3. Therefore, $p(e)$ is a price selection. Let e^* be a strategy selection such that $e^* = \mathbf{s}^*(m, \mathbf{h}^0)$ and let $\mathbf{h}^{1^*}$ be such that $\mathbf{h}^{1^*}(m) = e^*$. We have that $\mathbf{h}^2(\mathbf{s}^*) = \mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^{1^*})$ as $\mathbf{a}^0(\mathbf{s}^*) = \mathbf{s}^{0^*}(\cdot, \mathbf{h}^0) = \mathbf{h}^{1^*} = \mathbf{a}^0(\mathbf{s}^*|\mathbf{h}^{1^*})$ and $\mathbf{a}^1(\mathbf{s}^*) = \mathbf{s}^{1^*}(\cdot, \mathbf{h}^{1^*}) = \mathbf{s}^*|\mathbf{h}^{1^*} = \mathbf{a}^1(\mathbf{s}^*|\mathbf{h}^{1^*})$. Then, it must be that $e^* > 0$ as $\bar{\mathbf{A}}(\mathbf{s}^*)$ is irreducible and $p(e^*) = p(\mathbf{h}^2(\mathbf{s}^*|\mathbf{h}^{1^*}) = p(\mathbf{h}^2(\mathbf{s}^*))$. But then, it is straightforward to verify that

$$\mathbf{x}(m, e^*, p(e^*)) = \mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^*)(m), p(\mathbf{h}^2(\mathbf{s}^*))).$$

Suppose that there exists a strategy $e \in \mathbf{E}(m)$ such that

$$u_m(\mathbf{x}(m, e, p(e))) > u_m(\mathbf{x}(m, e^*, p(e^*))).$$

Let $s(m)$ be a strategy of the monopolist such that $\mathbf{s}^{0^*} \setminus s(m, \mathbf{h}^0)(m) = e$. Then, we have that $p(e) = p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m)))$ by the same argument used before. But then, we have that

$$\mathbf{x}(m, e, p(e)) = \mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^* \setminus s(m))(m), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m)))).$$

Therefore, it must be that

$$\begin{aligned} &u_m(\mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^* \setminus s(m))(m), p(\mathbf{h}^2(\mathbf{s}^* \setminus s(m))))) = u_m \mathbf{x}(m, e, p(e)) \\ &> \mathbf{x}(m, e^*, p(e^*)) = u_m(\mathbf{x}(m, \mathbf{h}^2(\mathbf{s}^*)(m), p(\mathbf{h}^2(\mathbf{s}^*)))), \end{aligned}$$

a contradiction. Thus the set of subgame perfect allocations is a subset of the set of monopoly allocations. Hence, the set of monopoly allocations coincides with the set of subgame perfect allocations. ■

Proof of Proposition 9. Let $\mathbf{w}^i(m) > 0$. Suppose that $\int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. Then, we have that $\int_{T^i} \mathbf{x}^{0i}(t, p)\, d\mu = 0$ as $\mu(T^i) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. But then, we have that $\int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu > 0$, for each $p \in \Delta \setminus \partial\Delta$. Therefore, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is restricted to the codomain R_{++}. For each $x \in R_{++}$, there exists at least one $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$, by the same argument used in the proof of Proposition 5. Then, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is onto as its range coincides with its codomain. Therefore, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible if and only if it is one-to-one. Hence, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible if and only, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$. ■

Proof of Proposition 10. Suppose that $\mathbf{w}^i(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible. Let $\mathring{p}(e)$ be a function which associates, with each strategy $e \in \mathbf{E}(m)$, the price vector $p = p^{0i}(e_{ij}\mu(m))$, if E is triangular, and is equal to $\{0\}$, otherwise. Then, $\mathring{p}(\cdot)$ is the unique price selection as $\pi(e) = \{\mathring{p}(e)\}$, for each $e \in \mathbf{E}(m)$. ■

Proof of Proposition 11. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Moreover, suppose that $\bar{x}^1 = \mathbf{w}^1(m)$. Then, we have that $\int_{T_0} \mathbf{x}^{01}(t, p)\, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t, p)\, d\mu = 0$ as $\mu(T^2) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. Therefore, we must have that $0 \leq \bar{x}^1 < \mathbf{w}^1(m)$. Let $\bar{e} \in \mathbf{E}(m)$ be such that $\bar{e}_{12} = \mathbf{w}^1(m) - \bar{x}^1$ and let $\bar{p} = \mathring{p}(\bar{e})$. Then, we have that

$$\begin{aligned}&\bar{x}^1\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \bar{p})\, d\mu \\ &= (\mathbf{w}^1(m) - \bar{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \bar{p})\, d\mu = \mathbf{w}^1(m)\mu(m),\end{aligned}$$

as $\bar{p} = \mathring{p}(\bar{e})$. Moreover, $\bar{p}$ is the unique price vector such that

$$(\mathbf{w}^1(m) - \bar{x}^1)\mu(m) = \int_{T_0} \mathbf{x}^{01}(t, \bar{p})\, d\mu,$$

as the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Then, it must be that

$$\bar{x}^2\mu(m) + \int_{T_0} \mathbf{x}^{02}(t, \bar{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

by Proposition 3. But then, we have that

$$\bar{x}^2 = e_{12}\frac{\bar{p}^1}{\bar{p}^2},$$

as $\bar{p}$ is market clearing. Therefore, we conclude that

$$\bar{x} = \mathbf{x}(m, \bar{e}, \bar{p}) = \mathbf{x}(m, \bar{e}, \mathring{p}(\bar{e})).$$

Hence, the feasible complement of the offer curve of the atomless part, the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R^2_+ : x = \mathbf{x}(m, e, \mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, the set of the final holdings of the monopolist. ■

Proof of Proposition 12. Suppose, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that

$$\begin{aligned}&\mathbf{x}^1(m, \tilde{e}, \tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu \\ &= (\mathbf{w}^1(m) - \tilde{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu = \mathbf{w}^1(m)\mu(m),\end{aligned}$$

and

$$\begin{aligned}&\mathbf{x}^2(m,\tilde{e},\tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu \\ &= \tilde{e}_{12}\mu(m)\frac{\tilde{p}^1}{\tilde{p}^2} + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu,\end{aligned}$$

as $\tilde{p}$ is market clearing. Then, we have shown that $\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})) \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, we have that $u_m(\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})))$ is maximal in the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$ as $u_m(\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})) \geq u_m(\mathbf{x}(m,e,\mathring{p}(e))$, for each $e \in \mathbf{E}(m)$, and $\{x \in R^2_+ : x\mu(m)+\int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x = \mathbf{x}(a,e,\mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, by Proposition 11. ■

Proof of Proposition 13. Suppose that $\mathbf{w}^i(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Then, there is a unique price vector $\bar{p} = p^{0i}(\bar{x}^i)$ such that $\bar{x}^i = \int_{T_0} \mathbf{x}^{0i}(t,\bar{p})\, d\mu$, as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i \int_{T_0} \mathbf{x}^{0i}(t,\bar{p})\, d\mu + \bar{p}^j \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

by Walras' law. Then, it must be that $\bar{x}^j = \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu$, where $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Therefore, $\{x \in R^2_+ : x^j = h(x^i)\} \subset \{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta\setminus\partial\Delta\}$. Suppose now that $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Let $\bar{p}$ be such that $\bar{x} = \int_{T_0} \mathbf{x}^0(t,\bar{p})\, d\mu$. Then, we have that $\bar{p} = p^{0i}(\bar{x}^i)$ as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i\bar{x}^i + \bar{p}^j\bar{x}^j = \bar{p}^i \int_{T_0} \mathbf{w}^i(t) + p^j \int_{T_0} \mathbf{w}^j(t),$$

by Walras' law. Then, we have that $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Therefore, $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x^j = h(x^i)\}$. Hence, the graph of the function $h(\cdot)$, the set $\{x \in R^2_+ : x^j = h(x^i)\}$, coincides with the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, the offer curve of the atomless part. ■

Proof of Proposition 14. Suppose that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible and that the function $p^{0i}(\cdot)$ is differentiable. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium such that $\tilde{e} < \mathbf{w}^i(m)$ and let $\tilde{\mathbf{x}}$ be the corresponding monopoly allocation. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and the necessary Kuhn-Tucker conditions imply that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^i} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = 0.$$

Then, we have that

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right).$$

Moreover, we have that

$$h(x^i) = -p^{0i}(x^i)x^i + \int_{T_0} \mathbf{w}^j(t)\, d\mu.$$

Differentiating the function $h(\cdot)$, we obtain

$$\frac{dh(x^i)}{x^i} = -\left(\frac{dp^{0i}(x^i)}{dx^i}x^i + p^{0i}(x^i)\right).$$

At the monopoly allocation $\tilde{\mathbf{x}}$, we have that

$$\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e}) = \frac{dp^{0i}(\int_{T_0}\tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}\int_{T_0}\tilde{\mathbf{x}}^i(t)\, d\mu + p^{0i}(\int_{T_0}\tilde{\mathbf{x}}^i(t)\, d\mu),$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} = \frac{dp^{0i}(\tilde{e}_{ij}\mu(m))}{dx^i}\mu(m)$ and $\tilde{e}_{12}\mu(m) = \int_{T_0}\mathbf{x}^{0i}(t, p(\tilde{e}))\, d\mu$. Hence, we have that

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0}\tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

Proof of Proposition 15. Let $\mathbf{w}^i(m) > 0$. Suppose that the strategy $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. Then, we have that

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$. But then, we must have that

$$u_m(\mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e})) \geq u_m(\mathbf{x}(m, e, \mathring{p}(e)),$$

for each $e \in \mathbf{E}(m)$, as

$$u_m(\mathbf{x}(m, e, \mathring{p}(e)) = \mathring{p}(e)e_{ij},$$

by Assumption $2'$, for each $e \in \mathbf{E}(m)$. Therefore, the strategy $\hat{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. The converse can be straightforwardly proved by the same argument. Hence, a strategy $\hat{e} \in E(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$, if and only if it is a monopoly equilibrium, with respect to the same price selection. Suppose that the function $p^{0i}(\cdot)$ is differentiable. Let $\hat{e} \in \mathbf{E}(m)$ be a Pareto monopoly equilibrium. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and the necessary Kuhn-Tucker conditions imply that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

$\blacksquare$

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