A Non-cooperative Approach to Dynamic Bargaining*

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Abstract

Many negotiations (for instance, among political parties or partners in a business) are characterised by dynamic bargaining: current agreements affect future bargaining possibilities. We study such situations using bargaining games á la Rubinstein (1982), with the novelty that players can decide how much to invest, as well as how to share the residual surplus for their own consumption. We show that under certain conditions, there is a unique (stationary) Markov Perfect Equilibrium characterised by immediate agreement. Moreover, standard results in bargaining theory can be overturned. For instance, despite the complexity of the bargaining game, there are equilibrium strategies as in an ultimatum, where the responder does not consume anything. Also, a more patient proposer may consumes less than his opponent. Additionally, a higher discount factor for one player may decrease the MPE investment rates for both players. We study the effect of different rates of time preferences, intertemporal elasticities of substitution and rates of return on the equilibrium demands.

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1 Introduction

Several bargaining situations in the most diverse contexts can be represented as bargaining games with dynamic accumulation, that is, bargaining games in which parties can invest part of a surplus and the invested surplus, in turn, affects the size of future surpluses. For instance, partners in a business need to negotiate not only over how to split profits among themselves, but also over how much profit should be re-invested for the following production period. Countries may attempt to find agreements over environmental issues by taking into account the fact that current decisions can affect the state of the environment and therefore future bargaining possibilities.

Bargaining games which allow parties to make both investment and consumption decisions a sequential number of times are almost unexplored (a discussion of the related literature is in the next section). We study a bargaining model á la Rubinstein (1982), where two (risk-averse) players attempt to agree not only on how to split a surplus for their own consumption but also on how much to invest. The level of investment affects the future capital stock and consequently, the surplus available in the following bargaining stage. The problem is complex. Not only do parties need to solve a (potentially protracted) bargaining stage, but also a dynamic accumulation problem since the agreement they reach at a specific stage affects future bargaining possibilities.

Our framework captures long-run relationship among parties in a general term, since capital and investment in our model are not necessarily physical. For instance, the players can be inventors in a partnership, who need to agree not only on how to share the profit from selling a new product/technology for their own consumption, but also on how much to spend on advertising and/or how much to re-invest in follow-up or new projects. Advertising and R&D expenditures are the (non-physical) investments which affect the resources available to the partners for the future projects. The capital stock represents all the resources available to the players for generating innovations (e.g., the knowledge acquired from previous inventions and resources accumulated through advertising previous products, such as clientele).

We show that in our model there is a unique stationary Markov Perfect Equilibrium (MPE) characterised by immediate agreement however, different types of equilibria can arise. Haggling cannot be an equilibrium phenomenon in our framework because delays are not strategically desirable.[1] A delay in reaching a current agreement implies a delay in

[1] Games with incomplete information are often characterised by strategic delays, since uncertainty can be partially solved by waiting (see, e.g., Admati and Perry,

realising not only current mutually beneficial gains but also *all* future gains. Therefore, even if the current cost of a one-period delay is very small, the total cost of a rejection can be very high for both parties. Also, a player can always avoid a rejection by investing sufficiently. Although the equilibrium is generally unique, there are three types of outcome that can arise, depending on players' characteristics, rate of return and depreciation rate. First, when there are some frictions in the bargaining process (i.e., both players are sufficiently impatient) and the rate of return is sufficiently high (and/or the depreciation rate is sufficiently low), then both parties extract all the residual surplus (given the extreme consumption demands, we call this the dual ultimatum-like MPE). Second, when parties are sufficiently asymmetric, there is an hybrid version of the ultimatum-like MPE where only one party (the most patient) extracts all the surplus not invested (while the other makes concessions). Third, there is an (interior) MPE in which both parties leave a positive share of the residual surplus to the opponent.

In standard bargaining games (without investment), extreme demands are not typical, unless the bargaining procedure and/or players' impatience are also extreme (for instance, in ultimatum games and/or when players value any agreement reached in the future as badly as the disagreement outcome). In our model, extreme demands are possible (even in the most interesting case in which a bargaining round is shorter than a production stage) as long as responders can be compensated for accepting them. Compensation for accepting an extreme demand is better than a rejection if first there are some (not necessarily extreme) frictions in the bargaining process and the rate of return is sufficiently high (and/or the depreciation rate is sufficiently low). Under these conditions, the proposer can invest enough to be able to consume the entire residual surplus without facing a rejection.

In our model a key role is played by an enriched discounting structure. We not only assume that players' have (potentially) different rates of time preference but also that in the game there are (potentially) different time intervals across bargaining and production stages. The differential in time preferences has been known to lead to significantly different results from the case in which players are homogeneous (see, for instance, see Lehrer and Pauzner (1999) in the context of repeated games). The assumption of (potentially) different time intervals captures the feature that typically the production stage, in which the surplus is generated,

1987). Similarly for games which include some stochastic elements, as the size of the surplus (Merlo and Wilson, 1995) or the arrival time of a future surplus (Acharya and Ortner, 2013). For games with complete information, delay can arise when one party bargain with two or more other parties (e.g., Cai, 2000).

is longer than a bargaining round, that is the time to make a counter-offer.[2] Hence, one of the aims of the paper is to investigate the resulting (complex) interplay of incentives in the game. We find that while in bargaining theory a more patient proposer can extract a larger surplus, in our model this is not the case (when the production stage is sufficiently long). The patient party is forced to demand little to prioritise his investment plan and avoid rejections, (which can be very costly in a dynamic framework). Moreover, when a party becomes more patient, we may have expected that he would invest more, instead the MPE investment rates may decrease for *both* players (when there are asymmetries). Also, if a party becomes more patient, this may make a rival better off.[3]

1.1 Related literature

Muthoo (1999) is, as far as we know, the first paper with a focus on a repeated (non-cooperative) bargaining game with investment decisions in addition to the standard consumption decisions. Muthoo (1999)'s focus is on steady-state stationary subgame perfect equilibria (while ours is on MPE). As a result, the investment decisions are simplified since parties need to invest as much as it is necessary so as to have surpluses of the *same* size. Indeed, Muthoo's aim is to apply Muthoo (1995), which is a bargaining model without investment (where parties share an infinite number of surpluses with the same size) to the accumulation problem. In this sense, the problem of how much parties invest in a strategic framework remains open.[4]

Subsequently, Flamini (2012) investigates an initial bargaining problem with dynamic accumulation with two major restrictions: the number of bargaining stages (or accumulations problems) is *finite* and parameters of the model assume specific values. The aim of the paper is to develops an algorithm to define the (symmetric) subgame perfect equilibrium and to investigate the convergence property of such an equilibrium when the number of bargaining stages increases. In contrast, the current paper focuses on a general model of dynamic bargaining with potentially an infinite number of bargaining stages and possibly asymmetries between players (hence, one obvious complication is that it is not possible to apply backward induction).

[2] A similar discounting structure has been considered first by Muthoo (1995), which is reviewed in the next section. Flamini (2007, 2007a) study the effects of this enriched discounting structure within an agenda formation problem.

[3] This is in accordance with Sorger (2006) and Houba et al. (2000), although the economics underpinning the result is different (see footnote 19).

[4] Indeed, as explained in Muthoo (1999, p. 330): "The application of dynamic capital investment... needs much further work. In particular, the analysis of the Markov SPE of the model awaits characterization".

There are also two main strands of literature, namely, on the *hold-up* problem and on the *tragedy of the commons,* which are related to the problem considered in this paper. However, there are fundamental differences between these problems and our dynamic bargaining game with investment. In the hold-up problem, parties have the ability to make sunk investments that affect the size of a surplus, before bargaining over the division of such a surplus. Since the investor, who bears all the costs of the investment, cannot appropriate all the benefits, the resulting investment is lower than the efficient level. Typically, only one party is involved in the investment problem, moreover, the investment is once and for all (see, for instance, Gibbons 1992, Muthoo 1998, Gul 2001).[5] Differently, the focus of this paper is on parties who *jointly* and *repeatedly* need to agree on how much to invest and consume.

The second strand of literature, on the tragedy of the commons, considers different parties who can extract part of a surplus for their own consumption and the remaining surplus will affect the size available in the next period (see, for instance, Levhari and Mirman, 1980, Dutta and Sandaram, 1993). The tragedy of the commons rests in the fact that parties consume more than the efficient level and therefore the surplus extinguishes quickly (over-exploitation of natural resources is a classic example). Although the typical framework analysing the problem of the tragedy of the commons is a dynamic accumulation game, it does not include any negotiation (everyone can consume as much as he wishes, given the stock available). Two notable exceptions, which have introduced bargaining in these dynamic accumulation games are Houba et al. (2000) and Sorger (2006). In Houba et al. (2000) parties can potentially bargain forever (á la Rubinstein), but differently from our framework, they need only agree once, since this agreement will be ever-lasting. In contrast, Sorger (2006) is closer to our paper, since in each period parties can reach an agreement over the levels of consumption (Sorger, 2006, also allows for endogenous threat points), however, the bargaining process is simplified in that it is given by the maximisation of Nash products. We consider a fully non-cooperative bargaining approach and characterise (analytically for some cases) the strategic behaviour that arises in equilibrium and the incentives that players need to take into account when forming their strategies. We show that the interplay of forces in our framework can be significant different from Sorger (2006) and asymmetries can have important consequences in the solution of the problem.

The paper is organised as follows. In the next section we present the

[5] An exception is in Che and Sákovics (2004) where parties keep investing until an agreement has been reached, however, once this is struck, the game ends.

model. In section 3 we analyse the MPE; first in a simple case, in which the elasticity of substitution is equal to 2 (in section 3.1.1) then more generally with focus on interior solutions in section 3.1.2. The case of ultimatum-like MPE is in section 3.2. Some final remarks are made in section 4. Most of the proofs are contained in the Appendix.

2 The Model

We consider a two-player bargaining game in which bargaining and production stages alternate (and each stage can start only after the other has taken place). At the production stage, a surplus is generated according to the production function $F(k_t) = Gk_t$, where G is the constant gross rate of return and k_t is the capital stock at period t, with $t = 0, 1, ...$ Production takes place in an interval of time τ. Once the surplus, $F(k_t)$, is generated, the bargaining stage begins and players attempt to divide this surplus. The bargaining stage is a classic infinitely-repeated alternating-offer bargaining game (Rubinstein, 1982) with the novelty that a proposal includes an investment plan. That is, a proposal by player i is a pair (x_i, φ_i), where φ_i is the investment share proposed by i and x_i is the share demanded by i over the remaining surplus. Hence, if the proposal is accepted, the level of investment is $I_{it} = \varphi_i k_t$ and the consumption levels are $c_{it} = x_i(G - \varphi_i)k_t$ for the proposer, and $c_{jt} = (1-x_i)(G-\varphi_i)k_t$, for the responder, with $i, j = 1, 2$. The subscript t indicates the dependence on capital at time t, denoted by k_t, which is the state variable in the model. Both consumption and investment plans (c_{it} and I_{it}, for player i) are linear time-invariant function of the state variable. A discussion on this is postponed to the end of this section. Players' per-period utility function has a CES form[6]:

$$u_i(c_{it}) = \frac{c_{it}^{1-\eta}}{1-\eta} \text{ for } \eta < 1 \tag{1}$$

with $i = 1, 2$. Also after an acceptance, the bargaining stage ends and the output available at the beginning of the next bargaining stage (at $t + 1$) is $F(k_{t+1})$, which it is given by the investment level I_{it} and the capital remaining after depreciation, $k_{t+1} = I_{it} + (1 - \lambda)k_t$, where λ is the depreciation rate ($0 < \lambda \leq 1$).[7] Regardless of whether the responder

[6] To simplify the exposition we focus only on the case of $\eta < 1$. For $\eta \geqslant 1$, it can be shown that there are no stationary linear MPE strategies.

[7] Often for tractability, it is assumed maximum depreciation ($\lambda = 1$), see for instance Ljungqvist and Sargent (2000), p. 33. However, this is an unrealist assumption. The analytical solutions we can obtain in our framework do not rely on the maximum depreciation assumption.

accepts the proposal at t he becomes the next proposer.[8]

If there is a rejection, a counter-offer can be made only after an interval of time Δ. In a one-period disagreement, d, parties receive $u_i(d) = 0$. We assume that the capital stock remains unchanged, k_t. Players must agree on the division of the current surplus, $F(k_t)$, before proceeding to a new production stage.[9]

Player i's time preference is represented by his discount rate h_i (with $i = 1, 2$). Since intervals of time may have different lengths (Δ, τ), there are two (potentially) distinct discount factors in our model: the between-cake discount factor $\alpha_i = exp(-h_i\tau)$ which captures the friction between bargaining stages (that is, the production time τ) and the within-cake discount factor $\delta_i = exp(-h_i\Delta)$ that takes into account that friction within the bargaining stage (that is, the interval of time between a rejection and a new proposal, Δ). In the first period, at $t = 0$, a bargaining stage starts and the surplus available is Gk_0, by assumption.

Figure 1 represents a time line in a specific example of this game where in the first bargaining stage, a proposal is accepted after n rejections, while in the second bargaining stage we assume that players immediately agree to disinvest their capital (hence, in this particular example the game ends after two bargaining stages).[10]

The focus is on (stationary) MPE, where the Markov strategies specify players' actions for each time period t as a function of the state of the system at the beginning of that period, k_t. Moreover, the aim of our analysis is to derive linear (time-invariant) strategies, that is, linear rules describing the investment and consumption paths as linear functions of the state k_t. Often the linearity of the strategies is assumed for tractability or it can be justified by players' inability to elaborate more complex rules (see e.g., Houba et al., 2000 and Sorger 2006). In addition,

[8]Our results are robust to different bargaining procedures. However, we need to exclude bargaining procedures which overly simplify the strategic interactions in the game, for instance, the case in which the successful proposer in a bargaining stage is assumed to be the first mover in the following bargaining stage. Indeed, in this case it can be shown that, in equilibrium, the proposer would simply demand to consume the same share as in the Rubisteinian game.

[9]This is to capture the feature that many long-run relationships are based on the engagement of the two parties. In other words, inertia (in the sense that in disagreement the parties keep implementing the old agreement) is excluded. Also Britz et al. (2013) assume that no production take place during disagreement in their two-period model of the firm.

[10]Note that parties can disinvest their capital if they wish ($\varphi_i < 0$) and, at most, they can invest the entire surplus ($\varphi_i = G$, with $i = 1, 2$). An alternative, but equivalent, specification of the model would be to allow the investment to be a linear function of the surplus (rather than the capital stock), and still allow the players to disinvest their capital if they wish to do so.

Muthoo (1999) and Flamini (2012) show that in their bargaining games, the subgame perfect equilibrium strategies are linear (and stationary). Given that Flamini (2012) contains a finite-time simplified version of our dynamic bargaining model, this suggests that the linear strategies are a natural candidate for our game.

3 Characterisation of the MPE Strategies

Let $V_i(k_t)$ (respectively, $W_i(k_t)$) be the sum of discounted payoffs to player i as a proposer (responder) in an arbitrary MPE. Then the problem can be written in the following recursive form:

$$V_i(k_t) = \max\{V_i'(k_t), \delta_i W_i(k_t)\} \tag{2}$$
$$W_j(k_t) = \max\{W_j'(k_t), \delta_j V_j(k_t)\} \tag{3}$$

where $V_i'(k_t)$ and $W_j'(k_t)$ are the sums of discounted payoffs in case of an acceptance, that is,

$$V_i'(k_t) = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} u_i(x_i, \varphi_i, k_t) + \alpha_i W_i(k_{t+1}) \tag{4}$$
$$\text{s.t. } W_j'(k_t) \geq \delta_j V_j(k_t) \tag{5}$$
$$W_j'(k_t) = u_j(x_i, \varphi_i, k_t) + \alpha_j V_j(k_{t+1}) \tag{6}$$

with per-period utility as in (1) and consumption levels given by

$$c_{it} \equiv c_i(x_i, \varphi_i, k_t) = x_i(G - \varphi_i)k_t$$
$$c_{jt} = c_j(x_i, \varphi_i, k_t) = (1 - x_i)(G - \varphi_i)k_t$$

while, in case of a rejection, the sum of discounted payoffs in (2) and (3) become

$$V_i(k_t) = \delta_i W_i(k_t) \text{ and } W_j(k_t) = \delta_j V_j(k_t) \tag{7}$$

with the equation of motion given by

$$k_{t+1} = \begin{cases} r_i k_t & \text{if there is an acceptance} \\ k_t & \text{otherwise} \end{cases} \tag{8}$$

with $r_i = \varphi_i + 1 - \lambda$, for $i, j = 1, 2$ and $i \neq j$. The rate r_i in the equation of motion, (8) indicates the gross rate of growth in the capital stock (k_{t+1}/k_t) after i's proposal is agreed. Hence, it is given by the investment rate, φ_i, plus the non-depreciation rate, $1 - \lambda$. As a result, when net investment $\varphi_i - \lambda$ is zero, $r_i = 1$, while for a positive (negative) net investment, $r_i > 1$ ($r_i < 1$). It varies between $[0, l]$ where $l = G + 1 - \lambda$. In the rest of the paper, to simply the notation, we often refer to MPE investment strategies in terms of the gross rate of growth r_i, rather than the investment rate φ_i.

Problem (2)-(8) is a recursive constrained problem with a complex structure since not only does (4) have a recursive form, but the constraint in (5) embodies another recursive problem (via the value functions $W_j'(k_t)$ and $V_j(k_t)$). Although, generally such problems cannot be solved (see Stokey and Lucas, 1989, Ljungqvist and Sargent, 2000), we can characterise the properties of the equilibrium outcome and we can also obtain an analytical solution under certain conditions.

First of all, we show that in a stationary MPE, delays cannot be sustained, however, extreme demands are possible. The intuition is that not only is haggling never strategically profitable for a proposer, but he can always invest an appropriate amount of surplus so that a rejection is unprofitable for the responder. Indeed, the proposer can even demand the entire residual surplus without facing a rejection as shown below.

Lemma 1. *Assume that $\alpha_i l^{1-\eta} < 1$, for $i = 1, 2$. Delays cannot be sustained in equilibrium. Moreover, extreme demands, in which at least a proposer consumes all the residual surplus ($x_i = 1$), can be part of an MPE.*
Proof. In Appendix. ■

The condition $\alpha_i l^{1-\eta} < 1$ is necessary for the existence of an equilibrium (as shown the proof of Lemma 1) and will be assumed henceforth. Differently from the existing literature (e.g., Houba et al., 2000 and Sorger, 2006), in our framework it is possible to have extreme demands ($x_i = 1$) in a stationary MPE. Moreover as shown in 3.2 these are sustainable for realistic parameter constellations.

Following Lemma 1, we obtain that $V_i(k_t) = V_i'(k_t)$ and $W_j(k_t) = W_j'(k_t)$ for $i, j = 1, 2$. Moreover, the Lagrangian is

$$L_i(k_t) = V_i(k_t) - m_i(\delta_j V_j(k_t) - W_j(k_t)) \tag{9}$$

where $V_i(k_t)$ and $W_j(k_t)$ are in (4) and (6), m_i is the (non-negative) Kuhn-Tucker multiplier (equal to zero when the constraint is slack) and x_i (and φ_i) are the share consumed (and invested, respectively), with $i, j = 1, 2$ and $i \neq j$.

Given the linearity of the equilibrium strategies, the value functions have the same (linear) functional form as the per-period utility function. Therefore, we write the value functions in such a form with coefficients which are left to be determined, we then solve the optimisation problem and derive the correct values of such coefficients. Hence, let ϕ_i (and μ_i) be the undetermined coefficients in player i's value functions when he proposes (responds respectively), that is

$$V_i(k_t) \equiv \phi_i \frac{k_t^{1-\eta}}{1-\eta} \text{ and } W_i(k_t) \equiv \mu_i \frac{k_t^{1-\eta}}{1-\eta}$$

Then, the optimisation problem becomes

$$\phi_i \frac{k_t^{1-\eta}}{1-\eta} = \max_{\substack{x_i \in [0,1] \\ \varphi_i \in [-(1-\lambda), G]}} u_i(x_i, \varphi_i, k_t) + \alpha_i \mu_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{10}$$

$$\text{s.t. } \mu_j \geqslant \delta_j \phi_j \text{ with} \tag{11}$$

$$\mu_j \frac{k_t^{1-\eta}}{1-\eta} = u_j(x_i, \varphi_i, k_t) + \alpha_j \phi_j \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{12}$$

$$k_{t+1} = r_i k_t \tag{13}$$

with $r_i = \varphi_i + 1 - \lambda$, $i, j = 1, 2$ and $i \neq j$.

3.1 Interior MPE

To solve the optimisation problem (10)-(13), first, we express the controls and subsequently the payoff coefficients in terms of auxiliary variables ψ_i and Kuhn-Tucker multipliers m_i, with $i = 1, 2$. Then, we derive the equilibrium conditions to solve for ψ_i and m_i, using a fixed point argument. Let[11]

$$M = \left\{ (m_i, \psi_i) | m_i, \psi_i > 0, 0 < \left[l \left(1 - \frac{(1 + m_i^{1/\eta})}{\psi_i} \right) \right]^{1-\eta} < \min \left(\frac{\delta_j}{\alpha_j}, \frac{1}{\alpha_i} \right) \right. \\ \left. \text{for } i, j = 1, 2, i \neq j \right\}$$

The first order conditions of (9) with respect of x_i and φ_i are respectively

$$x_i = \frac{1}{1 + m_i^{1/\eta}} \tag{14}$$

$$\varphi_i = \frac{(\alpha_i \mu_i + m_i \alpha_j \phi_j)^{1/\eta} G - (1 + m_i^{1/\eta})(1 - \lambda)}{(\alpha_i \mu_i + m_i \alpha_j \phi_j)^{1/\eta} + 1 + m_i^{1/\eta}} \tag{15}$$

If there is an interior solution, it must be that $m_i > 0$ and the constraint (11) holds as an equality, $\mu_j = \delta_j \phi_j$, for any $i, j = 1, 2$ and $i \neq j$. By the complementary slackness condition, if the constraint (11) is not binding, instead, the multiplier m_i is zero. The cases of $m_i = 0$ (with $i = 1$ or 2 or both) hold under certain conditions and are considered in section 3.2.

[11] A pair (m_i, ψ_i), with $i = 1, 2$ in M characterise a real and positive MPE proposal. Moreover the transversality condition is satisfied.

Using (14) and (15), the Bellman equation (10) can be re-written as

$$\phi_i = l^{1-\eta} \frac{1 + \alpha_i \mu_i g_i^{(1-\eta)/\eta}}{\psi_i^{1-\eta}} \tag{16}$$

where

$$g_i = \alpha_i \mu_i + m_i \alpha_j \phi_j \tag{17}$$
$$\psi_i = g_i^{1/\eta} + 1 + m_i^{1/\eta} \tag{18}$$

while equation (12) becomes

$$\mu_j = l^{1-\eta} \frac{m_i^{\frac{1-\eta}{\eta}} + \alpha_j \phi_j g_i^{(1-\eta)/\eta}}{\psi_i^{1-\eta}}$$

Using the constraint $\mu_j = \delta_j \phi_j$, the coefficients μ_j, ϕ_j can be written as in

$$\phi_i = \frac{l^{1-\eta}}{\psi_i^{1-\eta} - \alpha_i \delta_i l^{1-\eta} g_i^{\frac{1-\eta}{\eta}}} \text{ and } \mu_i = \frac{l^{1-\eta} \delta_i m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} \tag{19}$$

with $i, j = 1, 2$ with $i \neq j$. This implies that, the indifference condition $\mu_j = \delta_j \phi_j$ can be written as in (21), while from the definition of the auxiliary variable, that is,

$$\psi_i = (\alpha_i \mu_i + m_i \alpha_j \phi_j)^{1/\eta} + 1 + m_i^{1/\eta}$$

we obtain the following:

$$g_i = l^{1-\eta} \left(\frac{\alpha_i \delta_i m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} + \frac{\alpha_j m_i}{\psi_j^{1-\eta} - \alpha_j \delta_j l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} \right) \tag{20}$$

$$\frac{m_j^{\frac{1-\eta}{\eta}}}{\psi_j^{1-\eta} \delta_i - \alpha_i l^{1-\eta} g_j^{\frac{1-\eta}{\eta}}} = \frac{1}{\psi_i^{1-\eta} - \alpha_i \delta_i l^{1-\eta} g_i^{\frac{1-\eta}{\eta}}} \tag{21}$$

where $g_i = \left(\psi_i - (1 + m_i^{1/\eta}) \right)^{\eta}$. Then, if there is a solution $(\psi_i, m_i) \in M$ to the system (20) and (21), this defines the value function coefficients, μ_i and ϕ_i in (19), and the MPE shares consumed, (14), and invested:

$$r_i = l \left(1 - \frac{(1 + m_i^{1/\eta})}{\psi_i} \right) \tag{22}$$

3.1.1 An Example with Symmetric Players (and $\eta = 1/2$)

In this section, we assume that players are symmetric and the intertemporal elasticity of substitution is equal to 2 (i.e., $\eta = 1/2$). In this case, the game can be solved analytically, as shown in the following proposition.[12]

Proposition 1 *For $\eta = 1/2$, $h_i = h$ (i.e., $\delta_i = \delta$, $\alpha_i = \alpha$ for $i = 1, 2$), if $\alpha^2 l < 1$, there is a unique symmetric equilibrium in which each player successfully proposes the following consumption and investment plans:*

$$x = \frac{1}{1+m^2} \tag{23}$$

$$r = l\left(1 - \frac{(1+m^2)}{\psi}\right) \tag{24}$$

where

$$\psi = \frac{(1+m^2)(1-\delta m)^2\alpha^2 l}{\alpha^2 l(1-\delta m)^2 - (m-\delta)^2} \tag{25}$$

$$m = \frac{-(1-\delta^2)(1+\alpha^2 l) + \Gamma^{\frac{1}{2}}}{2\delta(1-\alpha^2 l)} \tag{26}$$

with

$$\Gamma = \left[(1+\delta^2)(1+\alpha^2 l)\right]^2 - 2^4\alpha^2 l\delta^2$$

Hence the MPE payoff coefficients are as follows:

$$\phi = \frac{l^{1/2}}{\psi^{1/2} - \alpha\delta l^{1/2}\left(\psi - (1+m^2)\right)^{1/2}} \tag{27}$$

$$\mu = \frac{l^{1/2}\delta m}{\psi^{1/2}\delta - \alpha l^{1/2}\left(\psi - (1+m^2)\right)^{1/2}} \tag{28}$$

Proof. in Appendix. ■

It can be shown that a proposer consumes more than a responder in a given round (in particular, $x > 1/2$ for $\delta \in [0, 1)$). Typically the investment level is inefficient, unless, bargaining is frictionless, as shown below.

Corollary 1. *For $h_1 = h_2$, at the limit for Δ that tends to 0, the stationary MPE is socially optimal:*

$$\lim_{\Delta \to 0} x = \frac{1}{2} \tag{29}$$

$$\lim_{\Delta \to 0} r = (\alpha l)^2 \tag{30}$$

[12] Another potentially simple case would be for $\eta = 1$. However, it can be shown that generally there is no time-invariant linear MPE for $\eta \geqslant 1$.

For $\Delta > 0$, bargaining leads to underinvestment.

Proof. At the limit for Δ that tends to 0, the multiplier m in (26) tends to 1, and therefore the consumption share x in (23) tends to 1/2. Moreover, given that ψ tends to $2(1-\alpha^2 l)^{-1}$, then, r in (24) goes to $(\alpha l)^2$, hence (30). It can be shown that when players are symmetric, a social planner, who maximises the sum of players' discounted payoffs, would invest a share $r = (\alpha l)^{1/\eta}$ (this is in line with the efficient consumption path derived in (45)) and would split the remaining surplus equally among the two parties (hence x and r, as in (29) and (30), are socially optimal). For $\Delta > 0$, r in (24) is lower than (30). Hence, there is underinvestment. ■

That is, frictionless bargaining can be efficient. Players with the same rate of time preference consume half of the residual surplus and invest a non-negative amount of surplus if sufficiently patient (i.e., $\alpha \geq (1-\lambda)^{1/2}/l$) otherwise parties disinvest as a social planner would efficiently choose to do.[13] When there are frictions in the bargaining stage, the equilibrium investment is inefficient. This result is driven by the same incentives as in the hold-up problem. Although in our framework a proposer can use a higher investment rate to facilitate an acceptance, he never needs to invest more than the socially optimal rate.[14]

Differently from Muthoo (1995, 1999), where the ultimatum equilibrium ($x = 1$) is sustainable for $\alpha > \delta$, in our dynamic model there will be always an interior solution (for $\eta = 1/2$) for any $\alpha \gtreqless \delta$, with $\delta > 0$. Indeed, it is straightforward to see that the equilibrium demands (23) and (24) are always interior for any value of discount factor $\alpha, \delta \in (0, 1)$.[15]

In the rest of this subsection we highlight the effect of patience on the MPE division in this simple case of symmetric parties. We show that the two discount factors interact in an interesting manner.

Corollary 2. *The MPE consumption demand x, in (23), is decreasing with the within-cake discount factor δ and increasing with the between-cake discount factor α, while the MPE investment r, in (24), is increasing with both δ and α.*

Proof. in Appendix ■

[13] This result is in accordance with Lockwood and Thomas (2002), which shows that the level of cooperation among players tends to the efficient level in the limit as players become patient, although their framework is quite different from ours, since players cannot bargain (moreover, they cannot reverse their actions, while in our model, parties are allowed to disinvest, $\varphi < 0$).

[14] This result can be generalised for other values of η.

[15] Only at the limit for δ that tends to 0, the equilibrium strategies are as in an ultimatum (that is, $x_i = 1$ and $r_i = 0$, see also footnote (28)).

As is typically the case in the Rubisteinian game, in our dynamic framework, an increase in the within-cake discount factor, δ, reduces the share demanded x, since counter-offers can be made more quickly. Interestingly, in dynamic bargaining, the between-cake discount factor, α, increases (rather than decreases, as for δ) the share demanded. This is because generally more patience implies higher investment, which allows a proposer to exploit the trade-off between current and future consumption. In particular, when the between-cake discount factor, α, increases, a proposer invests more and this allows his consumption demand x to increase without fearing a rejection. Instead, when it is the within-cake discount factor, δ, to increase, the exploitation of such a trade-off by a proposer does not have a dominant effect (since it is less costly to make counter-offers).

Although it is intuitive that more patient players invest more (r increases with patience), not all players pay the cost of a higher investment. As shown in the following corollary, although, generally, current consumption levels decrease with patience, when the within-cake discount factor, δ, increases, the cost of a higher investment level, is paid mainly by the proposer (his per-period consumption, $(G-\varphi)x$ decreases).

Corollary 3. *Generally, for a given k_t, in equilibrium, each party's per-period consumption level decreases when either discount factor increases, except the responder's consumption level, $(G-\varphi)(1-x)$, which instead increases with δ.*
Proof. in Appendix. ■

The intuition is that since more patient players invest more, the (per-period) residual surplus decreases, however, with diminishing frictions in the bargaining stage (δ increases), the responder can extract a larger share $(1-x)$ and a larger (current) consumption levels, $(G-\varphi)(1-x)$, for a given k_t. In other words, the prospect of larger future surpluses, in this case, are not sufficient to compensate a responder, his current consumption must increase.[16]

In the next section we focus on the complex interplay of forces in the more general case of possibly asymmetric players.

3.1.2 The General Case

We have solved system (20) and (21) for different values of the parameters. A selection of the numerical results is presented in the following

[16] In terms of overall payoffs, it is also possible to show that an increases in the within-cake discount factor, δ, can make a proposer worse off (ϕ decreases), for δ sufficiently high.

figures and tables[17], while the properties of the equilibrium are highlighted in the following remarks. The aim of the section is to highlight the effects of the complex interplay of the discounting structure, the rate of return G and the depreciation rate λ (via $l = G+1-\lambda$) on the MPE. Typical results which hold in bargaining theory can be overturned as shown next. In the first remark, the focus is on the share consumed (x_i for $i = 1, 2$).

Remark 1. *The more patient party consumes more than his opponent, unless l is sufficiently large and the production stage is sufficiently long.*

Table 1 and figure 2 are used to illustrate the remark. Table 1 presents an overview of the MPE demands for a simple range of discount factors (where all discount factors, δ_i, α_i, with $i = 1, 2$, can vary unconstrained in $(0, 1)$). In the case of symmetric players, we find symmetric MPE demands, (x, r), presented in the diagonal. The table entries for the asymmetric cases include player i's demands (x_i, r_i), in the first line, followed by player j's (x_j, r_j), in the second line, with $i, j = 1, 2$ and $i \neq j$.

Table 1 shows that the more patient proposer (j in this case) is able to consume a larger share (that is, in the first column within each non-diagonal table entry, the first value, x_i is lower than the second x_j). This is in line with standard bargaining theory (without investment), however, we show next that the possibility of investing overturns this result, when l is sufficiently large and the production stage is long (see figure 2).

Figure 2 shows the MPE demands as the rate l increases, for long production stages.[18] When l is sufficiently large (i.e., $l \geqslant 1.7$), the most patient proposer (j) demands a share *smaller* than his rival's ($x_j < x_i$). The intuition is that for high rates of return G and/or low depreciation rates λ, both parties have incentives to invest more, however, the most patient party wishes to invest significantly more than his opponent (the gap between r_j and r_i increases with l). Then, to prioritise investment, which is the variable affecting future bargaining possibilities, the more patient party must give up some of his current consumption while the impatient party can increase his demand (figure 2 shows that x_j decreases with l, while x_i increases with l).

[17] Unless otherwise specified, our discount factors are consistent (that is, once three of the discount factors are fixed, say, α_i, δ_j and δ_i, the fourth is uniquely defined, $\alpha_j = exp(ln(\alpha_i)ln(\delta_j)/ln(\delta_i))$.

[18] A long production stage implies a relative large lag between the within- and between-cake discount factors $(\delta_i - \alpha_i)$, especially for the most impatient player (when the discount factors are consistent, see 17).

A crucial factor, aside a large rate l, is the difference in players' discount factors $(\delta_i - \alpha_i)$. When l increases, relatively patient players have incentive to invest more, however, given lemma 1, production is required to be sufficiently long (s.t. $\alpha_i < l^{-(1-n)}$). This is why the result relies on a relatively substantial difference between players' discount factors. With shorter production stages and relatively high level of patience, l would be required (by lemma 1) to be sufficiently small and in this case, the most patient party would not be forced to make large concessions to his opponent (i.e., $x_j > x_i$).

i / j	α_j= 0.3, δ_j= 0.4	α_j= 0.4, δ_j= 0.5	α_j= 0.4, δ_j= 0.6	α_j= 0.6, δ_j= 0.7	α_j= 0.8, δ_j= 0.9
α_i= 0.3, δ_i= 0.4	$0.883, 0.020$	$0.823, 0.032$ $0.897, 0.034$	$0.727, 0.039$ $0.912, 0.041$	$0.664, 0.082$ $0.931, 0.097$	$0.389, 0.221$ $0.971, 0.255$
α_i= 0.4, δ_i= 0.5		$0.842, 0.046$	$0.750, 0.052$ $0.862, 0.052$	$0.688, 0.096$ $0.891, 0.107$	$0.404, 0.227$ $0.953, 0.259$
α_i= 0.4, δ_i= 0.6			$0.778, 0.058$	$0.717, 0.097$ $0.813, 0.108$	$0.430, 0.223$ $0.907, 0.251$
α_i= 0.6, δ_i= 0.7				$0.760, 0.145$	$0.461, 0.249$ $0.883, 0.267$
α_i= 0.8, δ_i= 0.9					$0.633, 0.305$

Table 1: Player i's (j's) MPE demands (x_i,r_i) in the first (second) line (for asymmetic cases), for $\eta = 1/2$ and $l = 0.7$.

Next, the focus is on the MPE investment plans.

Remark 2. *The more patient party invests more than his opponent, unless $\alpha_i = \alpha_j$, for $\eta = 1/2$ with $i, j = 1, 2$,* and $i \neq j$.

We would expect that since the most patient party is the most concerned about future payoffs then he will invest more than his opponent. This is generally confirmed in our numerical analysis, for instance, in table 1 (where $r_i \leqslant r_j$, recall the values r_i and r_j are in the second column of each cell), and in figure 2. However, we identify an exception in table 1 which highlights an interesting effect of the discounting structure on the investment share (this is investigated further in table 2 and the next remark). When players have the same between-cake discount factors, but different within-cake discount factors (see table 1 for $(\alpha_i, \delta_i) = (0.4, 0.5)$ and $(\alpha_j, \delta_j) = (0.4, 0.6)$ and table 2 for $\eta = 1/2$), they invest the same share of the surplus despite their asymmetry in patience. Such a scenario can be contemplated in our model (apart from comparative statics) only if the common parameter α is re-interpreted as a probability of game continuation after an acceptance (while production is instantaneous, bargaining rounds still take time). This indicates a key role played by the differences in the between-cake discount factors in the determination of the MPE investment plans. Although such a clear cut effect is generally true for any value of α (and the other parameters of the model) when $\eta = 1/2$, it becomes more subtle for other values of η, different from $1/2$, as shown in table 2 below.

	$\eta = 1/2$	$\eta = 2/3$
$\alpha = 0.15$	$0.364, 0.0110$	$0.386, 0.0337$
	$0.707, 0.0110$	$0.728, 0.0337$
$\alpha = 0.35$	$0.369, 0.0596$	$0.403, 0.1198$
	$0.712, 0.0596$	$0.743, 0.1199$
$\alpha = 0.55$	$0.381, 0.1470$	$0.436, 0.2349$
	$0.724, 0.1470$	$0.771, 0.2351$
$\alpha = 0.75$	$0.408, 0.2722$	$0.506, 0.3701$
	$0.748, 0.2722$	$0.825, 0.3707$
$\alpha = 0.95$	$0.481, 0.4320$	$0.700, 0.5114$
	$0.809, 0.4320$	$0.940, 0.5153$

Table 2: Player i's (j's) MPE demands (x_i,r_i) in the first (second) line, for $\delta_i = 0.9$, $\delta_j = 0.95$ and $l = 0.7$.

In the next two remarks, we summarise the effects of a change in a party's level of patience on the MPE investment strategies (remark 3) and payoffs (remark 4).

The most interesting effects highlighted in the next remark are that when a party becomes more patient, both proposers may *reduce* their investment shares, moreover, the within-cake discount factors can have a *negative* effect on the investment shares.

Remark 3. *If a party becomes more patient, both parties invest more, only when sufficiently similar, otherwise they decrease their investment plans. Moreover, although generally both the between- and the within-cake discount factors have a positive effect on the MPE investment shares, with pronounced asymmetries, the within-cake discount factor can have a negative impact on the shares invested.*

Figure 3 presents the effect of a change in player j's patience on the MPE demands, for $\eta = 1/2$, $l = 1.1$, $(\alpha_i, \delta_i) = (0.8, 0.9)$. The x-axis represents the player j's within-cake discount factor, δ_j (α_j varies accordingly, see footnote 17). The effects on the consumption shares (x_i and x_j) are simple here, but with a caveat. When player j becomes more patient he can consume a larger share (x_j increases) while his opponent consumes less (x_i decreases), in line with standard bargaining theory. The caveat is that as shown in the following remark, again with a sufficiently large rate l (and longer production periods) this result can be overturned.

Figure 3 also shows that the investment shares are affected in a non-monotonic way by a change in player j's patience. Starting from low levels of patience for player j, while j becomes more patient he will *reduce* his investment share (r_j decreases) to consume more and his opponent is forced to reduce his investment share (r_j decreases) to make larger concessions to player j. However, as player j's level of patience increases further ($\delta_j \geqslant 0.4$), player j invests more and eventually his opponent also becomes willing to increase his investment share (δ_j is required to be larger than 0.5). This effect is interesting, because generally, we would expect that if a party becomes more patient, he would invest more (since his future payoffs are discounted less heavily) and subsequently his rival may be forced to invest more (as shown, for instance, in table 1, when we move along each row, r_i and r_j, the values in the second column of each cell, increase with player j's patience). Clearly, though, this is a key incentive only when players' asymmetry is mild. With more pronounced asymmetries, instead, more patience can imply a higher consumption share for an impatient party (to the extent that all players' investment shares are lowered).

We now disentangle the effect of the between-cake and within-cake discount factors on the investment levels (second part of remark 3). When players are symmetric, corollary 2 shows that both discount factors have a positive impact on the investment level. It is intuitive that when the future payoffs are discounted less heavily, parties invest more. However, we next show that the within-cake discount factor can have a *negative* impact on the MPE investment shares, when there is a pronounced asymmetry between players (in contrast with Corollary 2, which relies instead on the symmetry of the players). To see this, consider the last column in table 1 (for $\alpha_j = 0.8$) when player i's within-cake discount factor, δ_i, increases from 0.5 to 0.6 (while α_i remains unchanged). Then, both parties invest *less* (r_i and r_j decrease). The intuition is that an increase only in δ_i gives player i the ability to increase his (low) consumption level (by demanding a larger share, x_i, and by reducing his investment share). As a result, his more patient opponent is forced to make larger concessions (by reducing his investment and consumption plans). The pronounced asymmetry between players is crucial to obtain this result. With milder asymmetries, an increase in the impatient party's within-cake discount factor (δ_i) has the expected positive effect on parties' investment shares (in table 1, in the second column from the last, both r_i and r_j increase when $\alpha_j = 0.6$ and again δ_i increases from

0.5 to 0.6).

In accumulation games with simpler bargaining structures, it has been shown that, differently from standard bargaining theory, patience is weakness (see footnote 19). In our model, we can re-establish the result that patience is strength and demonstrate that patience can make a rival *better off*, under certain conditions, as shown in the following remark.

Remark 4. *Assume that player i is more impatient than j, but his patience increases. Then, his opponent, player j, consumes more and is overall better off, if l is sufficiently large and the production stage is sufficiently long.*

Figures 4a and b show the effect of an increase in player i's patience on the MPE demands (first panel) and payoffs coefficients (second and third panel) for $(\alpha_j, \delta_j) = (0.8, 0.95)$, when the production stage is relatively long (that is, a party's between-cake discount factor, say j's, α_j, is significantly lower than his within-cake discount factor, δ_j, see also footnote 18). In terms of parameter constellations, the only difference between figure 4a and 4b is that in the former l is lower ($l = 1.2$ in figure 4a and 1.3 in figure 4b). This is a crucial difference: while in figure 4a, we obtain the standard result that as a player (i) gets more patient, his rival is worse off (the value function coefficients, ϕ_j and μ_j decrease, see the middle panel of figure 4a), in figure 4b, on the contrary, the opponent (j) is *better off* (ϕ_j and μ_j increase for $\delta_i > 0.85$, see the middle panel of figure 4b).

To understand this result, we look into the effect of patience on the MPE strategies. In line with remark/figure 3, also in figure 4a and 4b, when player i's patience increases, both players invest more when the asymmetry in patience is less pronounced (both r_i and r_j first decrease then increase, see top panel in figures 4a and 4b). More interestingly, in contrast with figure 3, figure 4b shows that proposer j is able to consume a higher share (x_j increases) despite his opponent becoming more patient. In particular, while in figure 4a (as in figure 3), for relatively low l, x_j decreases monotonically, in figure 4b, with a higher return (in particular, $l = 1.3$), x_j increases with patience, for δ_i sufficiently large ($\delta_i > 0.88$). The key force behind this result is that at high level of patience, both investment rates are high and increasing (r_i increases for $\delta_i > 0.66$ and r_j increases for $\delta_i > 0.74$ in figure 4b), given the large rate of return ($l = 1.3$), future consumption level can compensate a relative higher demand by a patient player.[19]

[19] The result that "patience is not strenght" is also found in Houba et al. (2000) and Sorger (2006), although the mechanism behind this result is different. Since in

Finally, following Muthoo (1999, p. 307), we can interpret the value functions as a measure of *overall* bargaining power in long-run relationships.[20] Then, figures 4a and b show that, despite the concessions that a patient player must make in equilibrium, such a player maintains the highest bargaining power both as a proposer and as a responder (that is, $\phi_j > \phi_i$ and $\mu_j > \mu_i$, for any $\delta_i < 0.95 = 0.95$, see middle and bottom panels in figures 4a and b).

3.2 Ultimatum-like MPE

In this section, we focus on MPE strategies in which at least one player can extract all the surplus not invested. Given that such extreme demands are typically found in the standard ultimatum procedure, we name them ultimatum-like strategies. In the next section both players are able to consume all the residual surplus (this is the dual ultimatum-like MPE), while in the following section only one player is able to do so (this is the hybrid ultimatum-like MPE). Note that, generally the ultimatum-like MPE strategies and the interior MPE, identified in the previous sections, do not coexist.

3.2.1 Dual Ultimatum-like MPE

Let

$$b_i = \left[\alpha_i^\eta (\alpha_j l)^{1-\eta}\right]^{\frac{2}{2\eta-1}}$$

with $\eta \neq 1/2$ and[21]

$$D = \{(\alpha_i, \delta_i, l) | \alpha_i, \delta_i \in (0,1), l > 0, \delta_j \leq \alpha_j (lb_i)^{1-\eta}, b_i \in (0,1) \\ \text{for } i, j = 1, 2, i \neq j\}$$

Proposition 2 *If $\eta \in (1/2, 1)$ and $(\alpha_i, \delta_i, l) \in D$ there is a unique (linear) MPE in which each proposer consumes all the surplus not invested ($x_i = 1$) and his investment plan is*

$$r_i = lb_i \tag{31}$$

The value function coefficients are

$$\phi_i = \frac{l^{1-\eta}}{(1-b_i)^\eta} \text{ and } \mu_i = \frac{b_i^\eta}{(1-b_i)^\eta \alpha_i} \tag{32}$$

the disagreement phase of their models, parties can consume as much as they wish, the impatient player can strategically use the threat of a delay to obtain a better agreement. In our framework, instead, is the higher investment rate that makes player j better off (this is not detrimental to the player i).

[20] Interpersonal comparisons of expected utility are allowed.

[21] A triple (α_i, δ_i, l) in D characterises a real and positive MPE proposal, in which it is subgame perfect to accept extreme consumption demands. Moreover the transversality condition is satisfied. The last two constraints in explicit form are in (33) and (34).

for $i, j = 1, 2$ *with* $i \neq j$.

Proof. in Appendix. ■

Ultimatum-like MPE can exist in our model when players are relatively patient, because a proposer can compensate a responder, who has a zero per-period consumption, by investing sufficiently therefore increasing his future consumption levels (as a proposer). A key role is played by the elasticity of substitution, which is required to be sufficiently small ($\eta > 1/2$). The intuition is that the curvature of the utility function in (1) decreases when consumption levels increase, but decreases less when η is large. As a result, a player's utility is higher for any positive level of consumption when η is larger. Therefore, the higher η, the higher the compensation a player will obtain after accepting an ultimatum-like proposal. This is why for $\eta > 1/2$ the responder can defer consumption and accept an ultimatum-like proposal that gives him zero current consumption while for $\eta \leq 1/2$ responders accepting the ultimatum-like proposal would not be sufficiently compensated by future consumption levels.

The equilibrium path for investment under the ultimatum-like MPE is simpler than the one highlighted in Remark 3. The investment rates in (31) increase with players' patience. Therefore, if player i becomes more patient not only does he increases his investment but so does his opponent. This clear-cut effect is due to that fact that only the between-cake discount factors affect the size of r_i (see (31)).[22] Although as a result of a higher investment the current surplus available for consumption is reduced, the overall effect on player j's payoff is positive (using (32), it can be shown that the coefficients for j's value functions both as a proposer, ϕ_j, and as a responder, μ_j, are increasing with α_i).[23]

Moreover, in our framework players can easily obtain an acceptance by investing more, however, their investment shares do not need to be higher than the efficient level. Indeed, as for the interior MPE, also in the ultimatum-like MPE, there is underinvestment. That is, for $h_1 = h_2$, the investment rates in (31) are lower than the socially optimal rate $r = (\alpha l)^{1/\eta}$ (see proof of corollary 1, since $\alpha l^{1-\eta} < 1$).

It has been shown that in long-run relationships without dynamic accumulation (see Muthoo, 1995), players can have extreme forms of bargaining power where proposers consume all the residual surplus, although, only under unlikely conditions (in which the production stage is quicker than the length of a bargaining round, $\Delta \geq \tau$, see Muthoo

[22] The within-cake discount factors affects the solution only via its support D.

[23] See footnote 20. Note that the models in Houba et al. (2000) and Sorger (2006) do not support ultimatum-like MPE.

1995, p. 594). In our dynamic framework, an ultimatum-like MPE can be sustained under less restrictive conditions (that is $\Delta < \tau$) as long as the investment rates are sufficiently large ($r_i = lb_i \geq (\delta_j/\alpha_j)^{1/(1-\eta)}$) so as to compensate a responder for accepting extreme proposals. However, there must be some frictions in the bargaining stage. If, instead, counter-offers can be made instantaneously (the interval Δ tends to 0), extreme demands are not sustainable in equilibrium. To see this, we re-write the last two conditions in the feasibility set D as follows

$$\frac{\delta_j^{2\eta-1}}{\alpha_j^{1-2\eta+2\eta^2}\alpha_i^{2\eta(1-\eta)}} \leq l^{1-\eta} < \frac{1}{\alpha_i^{\eta}\alpha_j^{1-\eta}} \tag{33}$$

$$\delta_j < \left(\frac{\alpha_j}{\alpha_i}\right)^{\eta} \tag{34}$$

At the limit for Δ that tends to 0, condition (34) cannot hold for both $i, j = 1, 2$ with $i \neq j$. Intuitively, when counter-offers can be made quickly, extreme demands cannot be compensated by sufficiently large investment levels and, as a result, a proposer must make some concessions (and leave a positive share of the surplus to the responder).

To give an idea of the support for the ultimatum-like MPE in proposition 2, we consider the case of symmetric players, then, conditions (33) and (34), can be written as:

$$\delta \leq (\alpha l^{1-\eta})^{\frac{1}{2\eta-1}} < 1 \tag{35}$$

The first inequality in (35) ensures that the surplus generated is sufficiently large, so that fairly patient parties still accept consuming nothing when they are responders, while the second inequality in (35) ensures that the equilibrium payoffs are finite. Obviously, the conditions in (35) are less stringent when $\Delta \geq \tau$. For instance, if after an agreement another bargaining stage can start straightaway ($\tau \to 0$) then the upper bound for the discount factor δ is simply $l < 1$. Assume, instead, it is quicker to make a counter-offer than producing a surplus ($\Delta < \tau$). Then, for fairly patient players, say $(\alpha, \delta) = (0.8, 0.9)$, with $\eta = 2/3$, then, the constraints in (35) imply that l must belong to a specific interval: $[1.76, 1.95)$.

3.2.2 Hybrid Ultimatum-like MPE

As shown in lemma 1, in addition to the dual ultimatum-like MPE, there can be other corner solutions of the problem, in which only *one* player (say, 1) is able to make extreme offers (that is, $x_1 = 1$ while $x_2 < 1$). Intuitively, this requires sufficiently asymmetric players (so that the most impatient party accepts an extreme proposal and makes concessions to the opponent when proposing). Indeed, this is what we obtain. Next, we derive the system which characterises the equilibrium. We then present the numerical analysis.

The System Characterising the Hybrid MPE The Bellman equations are given by (10) with $x_1 = 1$ and with $x_2 < 1$. In other words, the constraint of the acceptance condition (11) is not binding for player 2 (i.e., $m_1 = 0$), while it must be binding for player 2 ($m_2 > 0$). Let $c_1 = (\alpha_1\delta_1 l)^{\frac{1-\eta}{\eta}}$ and[24]

$$C = \left\{ (\psi_2, m_2, c_1) | \psi_2 > 1 + m_2, m_2 > 0, \frac{\delta_2}{\alpha_2} \leq c_1 < l^{1-\eta}, \right.$$
$$\left. c_1\alpha_1^2 \left[\left(1 - \frac{1 + m_2^{1/\eta}}{\psi_2}\right) l \right]^{1-\eta} < 1 \right\}$$

where $\psi_2 = (\alpha_2\mu_2 + m_2\alpha_1\phi_1)^{1/\eta} + 1 + m_2^{1/\eta}$, as in (17). Using the first order condition of the Lagrangian (9), that is, (14) and (15) with $m_1 = 0$, we obtain

$$x_1 = 1 \text{ and } r_1 = \frac{l(\alpha_1\mu_1)^{1/\eta}}{(\alpha_1\mu_1)^{1/\eta} + 1} \tag{36}$$

$$x_2 = \frac{1}{1 + m_2^{1/\eta}} \text{ and } r_2 = l\left(1 - \frac{1 + m_2^{1/\eta}}{\psi_2}\right) \tag{37}$$

Then, after some manipulations, the equilibrium coefficients can be re-written as

$$\phi_1 = l^{1-\eta}(1 + (\alpha_1\mu_1)^{1/\eta})^\eta \tag{38}$$

$$\phi_2 = \frac{l^{1-\eta}}{\psi_2}\left[1 + \alpha_2\mu_2(\psi_2 - 1 - m_2^{1/\eta})^{1-\eta}\right] \tag{39}$$

$$\mu_1 = \frac{l^{1-\eta}}{\psi_2}\left[m_2^{\frac{1-\eta}{\eta}} + \alpha_1\phi_1(\psi_2 - 1 - m_2^{1/\eta})^{1-\eta}\right] \tag{40}$$

$$\mu_2 = \alpha_2\phi_2\left[\frac{l(\alpha_1\mu_1)^{1/\eta}}{1 + (\alpha_1\mu_1)^{1/\eta}}\right]^{1-\eta} \tag{41}$$

Given the indifferent condition for player 1, $\mu_1 = \delta_1\phi_1$, then (38) becomes

$$\phi_1 = \frac{l^{1-\eta}}{\left(1 - (\alpha_1\delta_1 l^{1-\eta})^{\frac{1}{\eta}}\right)^\eta} \tag{42}$$

[24] A triple $(\psi_2, m_2, c_1) \in C$ characterises a real and positive MPE proposal, in which player 1's consumption demand is extreme ($x_1 = 0$) but acceptable to player 2. Moreover the transversality condition is satisfied.

Hence, using (42) and $\mu_1 = \delta_1\phi_1$, r_1 in (36) becomes

$$r_1 = c_1^{\frac{1}{1-\eta}} \tag{43}$$

and (41) can be re-written as

$$\mu_2 = \alpha_2\phi_2 c_1$$

The latter in (39) implies

$$\phi_2 = \frac{l^{1-\eta}}{\psi_2^{1-\eta} - \alpha_2^2 l^{1-\eta} c_1(\psi_2 - 1 - m_2^{1/\eta})^{1-\eta}}$$

Now, both ϕ_2 and μ_2 are written in terms of ψ_2 and m_2 (while ϕ_1 and μ_1 are already solved for). The equilibrium ψ_2 and m_2 are given by the solution of the following system

$$\begin{cases} \psi_2 = (\alpha_2\mu_2 + m_2\alpha_1\phi_1)^{1/\eta} + 1 + m_2^{1/\eta} \\ \mu_1 = \delta_1\phi_1 \end{cases} \tag{44}$$

with ϕ_1, μ_1 and μ_2 as in (42), (40) and (41) respectively.

Analysis Before moving to the numerical analysis, it is worth to note that, differently from the dual ultimatum case, in the hybrid MPE, player 1's MPE proposal and payoff coefficients are unaffected by his opponent's patience (in particular, r_1 in (43), ϕ_1 in (42) and $\mu_1 = \delta_1\phi_1$ are all independent of both α_2, and δ_2). In the dual case, we found that the within-cake discount factor δ_2 did not affect the MPE solution (except via its support), this is confirmed also in the hybrid case (see players' MPE strategies, payoffs and characterising system, in (36)-(44)), where also the between-cake discount factor α_2 plays such limited role. Hence, generally, when player 2 becomes more patient, in this corner solution of the problem, player 1's plans remain unchanged. It is possible to show that, in line with the previous analysis, with a higher between-cake discount factor, player 2's increases his investment plan and, as a result, he will increase his consumption share (although, this leaves player 1's strategies and payoffs unchanged).

Since generally there is not an analytical solution to system (44), we now discuss the numerical solutions. The focus is on changes in player 1's patience (given the limited impact of player 2's patience on the MPE).

A first striking feature of the hybrid case is that although only player 1 asks for the entire residual surplus ($x_1 = 1$), player 2 is able to demand almost the same (x_2 is almost 1), despite being significantly more impatient than his opponent. To see this, consider figure 5, which shows the MPE demands (x_i, r_i) (in the top panel) and payoffs coefficients ϕ_i and μ_i (in the bottom panel) with $i = 1, 2$, while player 1 becomes more patient, for $\eta = 4/5$, $l = 1.8$, $(\alpha_2, \delta_2) = (0.44, 0.45)$. The share x_2 remains always very close to 1 (even for δ_2 approaching 1).[25] Typically in bargaining models without investment an impatient player consumes little, however, with a dynamic component, this effect is much weaker. Exactly because player 2 does not value future gains as much as player 1, he can compensate player 1 by investing a portion of the surplus and leaving him very little to consume. Such a compensation is sufficient also due to the relative low elasticity of substitution.

Figure 5 also shows that both players invest more (r_i increases, with $i = 1, 2$) as player 1's becomes more patient (and in line with previous results, the more patient invests more, $r_1 > r_2$). However, the increasing investment does not significantly affect player 2's MPE payoffs (ϕ_2 and μ_2 are almost perfectly flat in figure 5), instead, in the dual ultimatum-like MPE, a more patient player would make the rival better off. We, next, disentangle the effects of the between- and the within-discount factors (α_1, δ_1) on player 2's MPE payoffs and show that they can be significant and contrasting (as shown next in figures 6 and 7).

Figures 6 and 7 presents the results as in figure 5: the MPE demands (x_i, r_i) with $i = 1, 2$ (in the top panels) and payoffs coefficients ϕ_2 and μ_2 (in bottom panels)[26], with the difference that *only one* of player 1's discount factors increase (for $\eta = 4/5$, $l = 1.8$, $(\alpha_2, \delta_2) = (0.44, 0.45)$). In particular, in figure 6, the within-cake discount factor δ_1 increases and the between-cake discount factor α_1 is fixed (and equal to 0.8973, consistently with $\delta_1 = 0.9$), while in figure 7, the between-cake discount factor α_1 increases and the within-discount factor δ_1 is fixed (and equal to 0.9).

When the discount factor δ_1 increases (figure 6), player 2 demands less and invests more, as a result he is worse off (ϕ_2 and μ_2 decrease). Differently, when it is the discount factor α_1 to increase, player 2 is better off (ϕ_2 and μ_2 increase in figure 7), this results from a higher demand (x_2 increases) and almost unchanged investment (see figure 7).

The results in figures 5-7 can be replicated with other parameter constellations, however, it is worth to note some points regarding the support for the hybrid MPE. First, if player 2 was the more patient, there would be no feasible hybrid solution to system (44). Second, a bargaining

[25] It can be easily verified that there cannot be a dual-ultimatum solution for the set of parameters considered in figure 5 (since the inequality $\delta_1 \leq \alpha_1 r_2^{1-n}$ in D cannot hold).

[26] To focus only on player 2's MPE coefficients, player 1's, ϕ_1 and μ_1, are omitted in figures 6 and 7 (but they follow the same monotonic trends as in figure 5).

round must be almost as long as a production stage (assuming the more realistic scenario that it is quicker to make a counter-offer). To see this, note that the acceptance condition (that is, the inequality $\frac{\delta_2}{\alpha_2} \leq c_1$ in C) is more likely to hold when player 2's discount factors, α_2 and δ_2, are sufficiently close (hence, the most patient player's discount factors, α_1 and δ_1 are even closer). Third, high capital growth must be possible. In particular, the parameter l is required to be sufficiently large (since $\frac{\delta_2}{\alpha_2} \leq 1 < l^{1-\eta}$, see C). This, in conjunction with a sufficiently high discount factor α_2 (hence close to δ_2), allows that extreme demands ($x_1 = 1$) can be accepted by the impatient party (player 2), that is, his future gains are sufficiently high.

4 Final Remarks

The novelty of our framework is that it addresses the problem of dynamic accumulation within a bargaining game, following a fully non-cooperative approach. We have shown that when investment is introduced within a bargaining game, the interplay of the forces can be very complex and that various lessons from standard bargaining theory can be overturned. An additional novelty of our model is that extreme consumption shares can be an equilibrium phenomenon, even with modest frictions, since a proposer can invest enough to compensate a responder for accepting a proposal that gives him zero current consumption. Moreover, although generally the investment strategy can give a proposer the ability to obtain an acceptance without making large concessions to his opponent, players will never over-invest. Indeed, they will typically under-invest. Only frictionless bargaining can be efficient.

In addition, when players are asymmetric, they agree on *dynamically* inefficient divisions, since typically they will share the surplus not invested, instead it would be Pareto superior to let only the impatient player consume a positive share of the initial N surpluses and let only the patient party consume afterwards. Although the solution is dynamically inefficient, it is dynamically consistent. Dynamic consistency emerges from the fact that players do not commit to future behaviours, in each period they simply optimise their behaviour taking into account the effect of current decisions.

Institutions could re-establish efficiency. For instance, with focus on players with similar rates of time preference, if they could commit to share all the residual surpluses equally, then, even if impatient they can behave efficiently (regardless of the bargaining procedure adopted). Suppose for instance that before entering a business two partners could sign a contract that specifies that each will obtain half of the profits not re-invested. Then, their investment plan would be efficient. Accordingly, policy makers may wish to create institutions which guarantee an appropriate division of mutual gains to encourage efficient investment paths in ongoing negotiations.

APPENDIX

Proof of Lemma 1[27]
Consider any subgame where player i proposes first. Let k_t be the state variable and $V_i(k_t)$ ($W_j(k_t)$) the sum of discounted payoffs to player i (j) in an arbitrary MPE, with $i, j = 1, 2$ and $i \neq j$. The sum of discounted utilities $V_i(k_t)$ and $W_i(k_t)$ are bounded:

$$V_i(k_t), W_i(k_t) \in \left[0, \frac{k_t^{1-\eta}}{1-\eta} \frac{l^{1-\eta}}{\left(1 - (\alpha_i l^{1-\eta})^{1/\eta}\right)^{\eta}}\right]$$

if $\alpha_i l^{1-\eta} < 1$. The upper bound has been derived by assuming that the investment and consumption paths are to maximise player i's payoff as in a standard saving model (without bargaining). Using the value function iteration method it can be shown that the per-period consumption for player i is given by

$$l\left(1 - (\alpha_i l^{1-\eta})^{1/\eta}\right) k_t \tag{45}$$

and all the surplus not consumed by player i is invested. The condition $\alpha_i l^{1-\eta} < 1$ must hold to have the convergence of the sum of discounted payoffs.

Within the bargaining framework, if player j accepts a proposal (x_i, φ_i), when the state is k_t, he gets

$$u_j(x_i, \varphi_i, k_t) + \alpha_j V_j(k_{t+1})$$

while if he rejects it, he obtains $\delta_j V_j(k_t)$. Therefore, the proposal is accepted if and only if

$$u_j(x_i, \varphi_i, k_t) \geq \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{46}$$

We now distinguish three cases, in the first, the RHS of (46) is non-positive for both players, in the second is positive for both players and in the third is non-positive for only for one player.

[27] The proof generalises the arguments in Muthoo (1995) to the case of dynamic accumulation (and concave per-period utility). The proof holds for any concave per-period utility as long as the sum of discounted utility in the standard saving-consumption problem (without bargaining) are bounded. However, given the focus of the paper, in the following we assume CES per-period utility.

In the first case, where the RHS of (46) is non-positive for $i, j = 1, 2$ and $i \neq j$, in equilibrium player i consumes all the surplus not invested ($x_i = 1$, with $i = 1, 2$) without facing a rejection. Subgame perfections also requires that player i's investment maximises the sum of his discounted payoff ($V_i(k_t)$).

In the second case, if the RHS of (46) is positive for both players, then there are two possibilities: a proposer prefers to make either an acceptable offer or an unacceptable one. In the former, the optimal proposal (x_i, φ_i) must be such that the investment plan φ_i maximises the sum of his discounted payoff ($V_i(k_t)$) under the constraint (46) while the share x_i must be strictly smaller than 1 (otherwise the LHS of (46) is 0), but as large as possible so as to obtain an acceptance ((46) is satisfied). Hence, in this scenario the proposal (x_i, φ_i) is such that the following holds

$$u_j(x_i, \varphi_i, k_t) = \delta_j V_j(k_t) - \alpha_j V_j(k_{t+1}) \tag{47}$$

and this is the case when

$$u_i(x_i, \varphi_i, k_t) + \alpha_i W_i(k_{t+1}) \geq \delta_i W_i(k_t)$$

Alternatively, the proposal is unacceptable, that is,

$$u_j(x_i, \varphi_i, k_t) + \alpha_j V_j(k_{t+1}) < \delta_j V_j(k_t) \tag{48}$$

and this is the case when

$$u_i(x_i, \varphi_i, k_t) + \alpha_i W_i(k_{t+1}) < \delta_i V_i(k_t) \tag{49}$$

We now show that (47) must hold. Suppose, by contradiction, that it does not, then there are 2 cases: a and b. In case a, assume that (48) holds for both players. Then, there is no acceptable offer when the state is k_t (therefore, the state k_{t+1} is never reached) and $V_j(k_t) = V_j(k_{t+1}) = W_i(k_t) = W_i(k_{t+1}) = 0$ for any $i, j = 1, 2$, which lead to a contradiction. In case b, assume that (48) holds only for one player, without loss of generality, say 1. Then, player 1 makes an unacceptable offer, that is, (49) holds for $i = 1$ and $j = 2$ while player 2 makes an acceptable offer, that is (47) holds for $i = 2$ and $j = 1$. As a result, $V_1(k_t) = \delta_1 W_1(k_t) = \delta_1 W_1'(k_t)$, similarly, $W_2(k_t) = \delta_2 V_2(k_t) = \delta_2 V_2'(k_t)$,

$$\begin{aligned} V_2'(k_t) &= u_2(x_2, \varphi_2, k_t) + \alpha_2 W_2(k_{t+1}) \\ W_1'(k_t) &= u_1(x_2, \varphi_2, k_t) + \alpha_1 V_1(k_{t+1}) \end{aligned}$$

Since (47), for $i = 2$ and $j = 1$, is also equivalent to $W_1'(k_t) = \delta_1 V_1(k_t)$, then, it must be $W_1'(k_t) = W_1(k_t) = 0 = V_1(k_t)$, which is a contradiction.

In conclusion, if the RHS of (46) is positive for both players, for any state k_t, a proposal (x_i, φ_i) is part of an MPE if immediately accepted.

Finally, in the third case, without loss of generality, the RHS of (46) is non-positive for $j = 1$ only. Then, in equilibrium player 2 must consume all the surplus not invested ($x_2 = 1$) without facing a rejection, while player 1, if he makes an acceptable offer, it must be that $x_1 < 1$ and such that (46) is binding, with $i = 1$ and $j = 2$. We now show that player 1 does not have any incentive to make an unacceptable offer. Assume, by contradiction that player 1 is better off in making an unacceptable offer, when the state is k_t,

$$u_1(x_1, \varphi_1, k_t) + \alpha_1 W_1(k_{t+1}) < \delta_1(0 + \alpha_1 V_1(k_{t+1})) \tag{50}$$

If player 1 would set $x_1 = 0$ and $\varphi_1 = \varphi_2$, when player 2 would accept this offer and player 1 would get $\alpha_1 W_1(k_{t+1})$ which, using the acceptance condition for (x_2, φ_2), would be not smaller than $\delta_1 \alpha_1 V_1(k_{t+1})$, hence player 1 would be better off, which is a contradiction.

Proof of Proposition 1

For $\eta = 1/2$ and $h_i = h$, system (20)-(21) has a unique symmetric solution which is given by (25) and (26). Then, the MPE demands (23) and (24) and payoff coefficients (27) and (28) follow from (14), (19) and (22). Moreover the support set M can be replaced by the condition $\alpha^2 l < 1$, given that $\alpha, \delta \in (0, 1)$ and $l > 0$.

Proof of Corollary 2

The multiplier decreases with α and increases with δ, because,

$$\frac{\partial m}{\partial \alpha} = \frac{2l\alpha(1-\delta^2)\left((1+l\alpha^2)(1-\delta^2) - \Gamma^{\frac{1}{2}}\right)}{\delta(1-l\alpha^2)^2\Gamma^{\frac{1}{2}}} < 0$$

$$\frac{\partial m}{\partial \delta} = \frac{(1+\delta^2)(1+l\alpha^2)\left(\Gamma^{\frac{1}{2}} - (1+l\alpha^2)(1-\delta^2)\right)}{2(1-l\alpha^2)\delta^2\Gamma^{\frac{1}{2}}} < 0$$

for any α, δ in $(0, 1)$ and $l\alpha^2 < 1$. Therefore the MPE demand (23) increases with α and decreases with δ.

For the MPE investment (24),

$$sgn\left(\frac{\partial \varphi}{\partial \alpha}\right) = -sgn\left(\frac{\partial((1+m^2)/\psi)}{\partial \alpha}\right) = sgn\left(\frac{\partial \frac{(\delta-m)^2}{(1-\delta m)^2\alpha^2 l}}{\partial \alpha}\right)$$

where

$$\frac{\partial \frac{(\delta-m)^2}{(1-\delta m)^2\alpha^2 l}}{\partial \alpha} = \frac{8(1+\delta^2+l\alpha^2(1-3\delta^2) - \Gamma^{\frac{1}{2}})(1+\delta^2)(1-l\alpha^2)z}{\Gamma^{\frac{1}{2}}(3-\delta^2-l\alpha^2(1+\delta^2) - \Gamma^{\frac{1}{2}})^3 l\alpha^3\delta^2} \tag{51}$$

with

$$z = 1 + \delta^2 + l\alpha^2(1 + \delta^4 - 6\delta^2 + l\alpha^2\delta^2(1 + \delta^2)) - \Gamma^{\frac{1}{2}}(1 - \delta^2 l\alpha^2)$$

It can be shown that all terms in (51) are positive. Therefore the MPE investment (24) increases with α.

Finally,

$$sgn\left(\frac{\partial\varphi}{\partial\delta}\right) = sgn\left(\partial\left(\frac{\delta - m}{1 - \delta m}\right)^2 / \partial\delta\right)$$

where

$$\frac{\partial\frac{\delta - m}{1 - \delta m}}{\partial\delta} = \frac{4(1 - \delta^2)(1 + l\alpha^2)w}{\Gamma^{\frac{1}{2}}(\Gamma^{\frac{1}{2}} - 3 + \delta^2 + l\alpha^2(1 + \delta^2))^2\delta^2} \tag{52}$$

with

$$w = (1 + \delta^2)(1 + \alpha^4 l^2\delta^2) + l\alpha^2((1 - \delta^2)^2 - 4\delta^2) - \Gamma^{\frac{1}{2}}(1 - \delta^2 l\alpha^2)$$

It can be shown that all the terms in (52) are positive, then the MPE investment (24) increases with δ.

Proof of Corollary 3

From corollary 2 we can conclude that an increase in δ decreases the proposer's consumption, $(G - \varphi)x$, (since x decreases while φ increases with δ) and an increase in α decreases the responder's consumption, $(G-\varphi)(1-x)$ (since x and φ increases with α). In addition, the proposer's current consumption is as follows,

$$(G - \varphi)x = l/\psi = \frac{2l(1 - l\alpha^2)^2}{q + \Gamma^{\frac{1}{2}}(1 + l\alpha^2)}$$

where

$$q = (1 + \delta^2)(1 + la^2)^2 - 8la^2$$

It can be shown that $\partial\psi/\partial\alpha > 0$, then we can conclude that $(G-\varphi)x$ decreases with patience. Instead, the responder's per-period consumption, which can be written as $\frac{lm^2}{\psi}$, increases with δ.

Proof of Proposition 2

By the complementary slackness condition, if the constraint $W_j \geq \delta_j V_j$ is not binding, the multiplier m_i in (9) is zero. Then, the first order conditions (14) and (15) for $m_i = 0$ become:

$$\varphi_i = \frac{(\alpha_i\mu_i)^{1/\eta}G - (1 - \lambda)}{(\alpha_i\mu_i)^{1/\eta} + 1} \tag{53}$$

$$x_i = 1 \tag{54}$$

We now input the first order condition in the Bellman equation (10) and after simplifying we obtain that $\phi_i = \psi_i^\eta l^{1-\eta}$ with $\psi_i = 1 + (\alpha_i \mu_i)^{1/\eta}$. The rate of investment (**??**) can now be written as $\varphi_i = G - \frac{l}{\psi_i}$ or $r_i = l\left(1 - \frac{1}{\psi_i}\right)$. Consequently, the responder's MPE payoff coefficient is

$$\mu_j = \alpha_j \phi_j r_i^{1-\eta}$$

This and the definition of ψ_i, that is $(\psi_i - 1)^\eta = \alpha_i \mu_i$, implies the following system:

$$\alpha_j^2 l^{2(1-\eta)} \left(\frac{\psi_i - 1}{\psi_i}\right)^{1-\eta} = \left(\frac{\psi_j - 1}{\psi_j}\right)^\eta$$

For $\eta \in (1/2, 1)$, there is a unique solution given by

$$\psi_i = \frac{1}{1 - b_i} \tag{55}$$

with $b_i \in (0, 1)$.[28] This defines an acceptable offer if the responder is better off in accepting rather than rejecting the offer:

$$\alpha_j \phi_j l^{1-\eta} \left(1 - \frac{1}{\psi_i}\right)^{1-\eta} \geq \delta_j \phi_j$$

that is, $\alpha_j (l b_i)^{1-\eta} \geq \delta_j$ for $i, j = 1, 2$ with $i \neq j$. The latter, together with $b_i \in (0, 1)$, implies the conditions set in (33) and (34). Under such conditions, also the transversality condition is satisfied (since $\alpha_i^2 (l^2 b_i b_j)^{1-\eta} < 1$ implies the second inequality in (33)). Finally, given (55), the coefficients $\phi_i = \psi_i^\eta l^{1-\eta}$ and $\mu_i = (\psi_i - 1)^\eta / \alpha_i$ can be written as in (32).

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[28] For $\eta \in (0, 1/2]$, the condition $\psi_i > 1$ cannot hold. This implies that capital would be fully disinvested ($x_i = 1$, $r_i = 0$). This can be part of an MPE strategies if and only if the frictions in the bargaining stage, Δ, go to $+\infty$ (or players are infinitely impatient). To see this, note that the left hand side of (46) must be non-negative if $x_i = 1$, that is $\delta_j - \alpha_j r_i^{1-\eta} \leq 0$. Hence, if $r_i = 0$, it must be that $\delta_j = 0$, for any $i, j = 1, 2$ and $i \neq j$.

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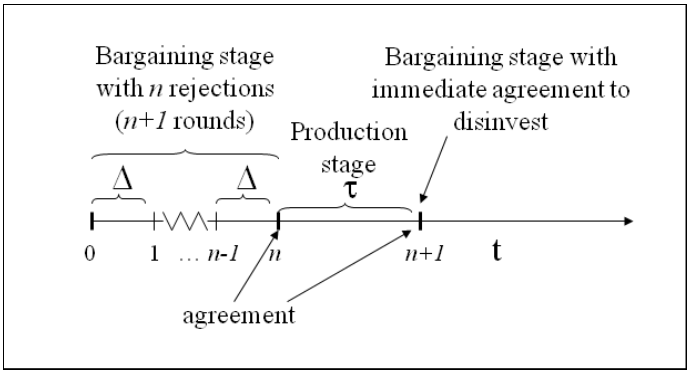

Figure 1. Time line for a game with n (0) rejections in the first (second, respectively) bargaining stage.

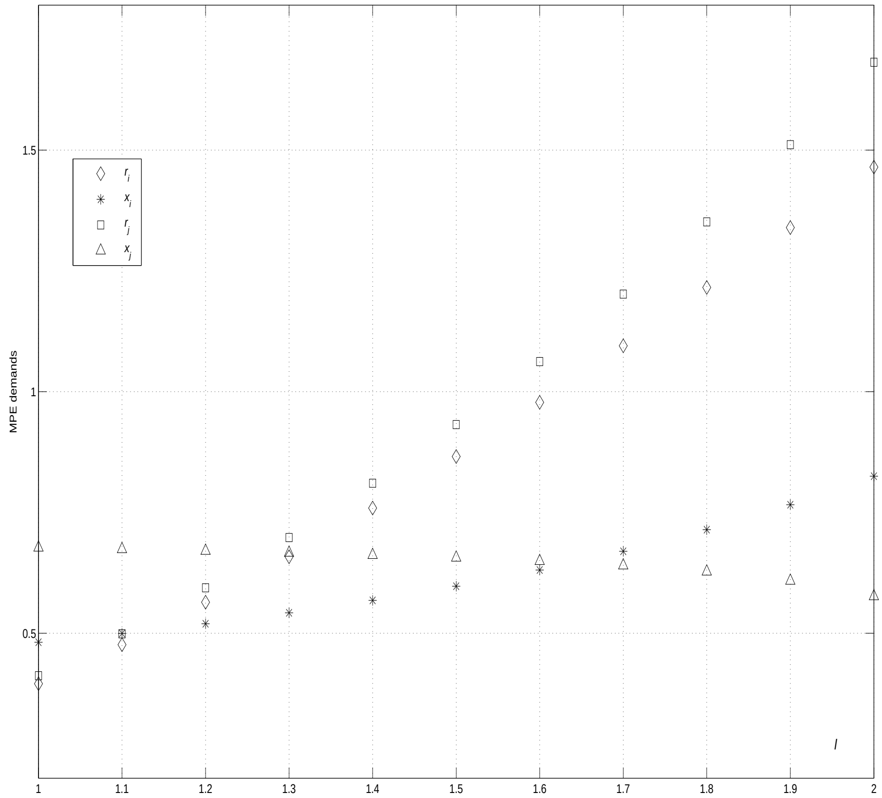

Figure 2. MPE demands for η=1/2, (α_i, δ_i)=(0.5,0.9) and (α_j, δ_j)=(0.7,0.95).

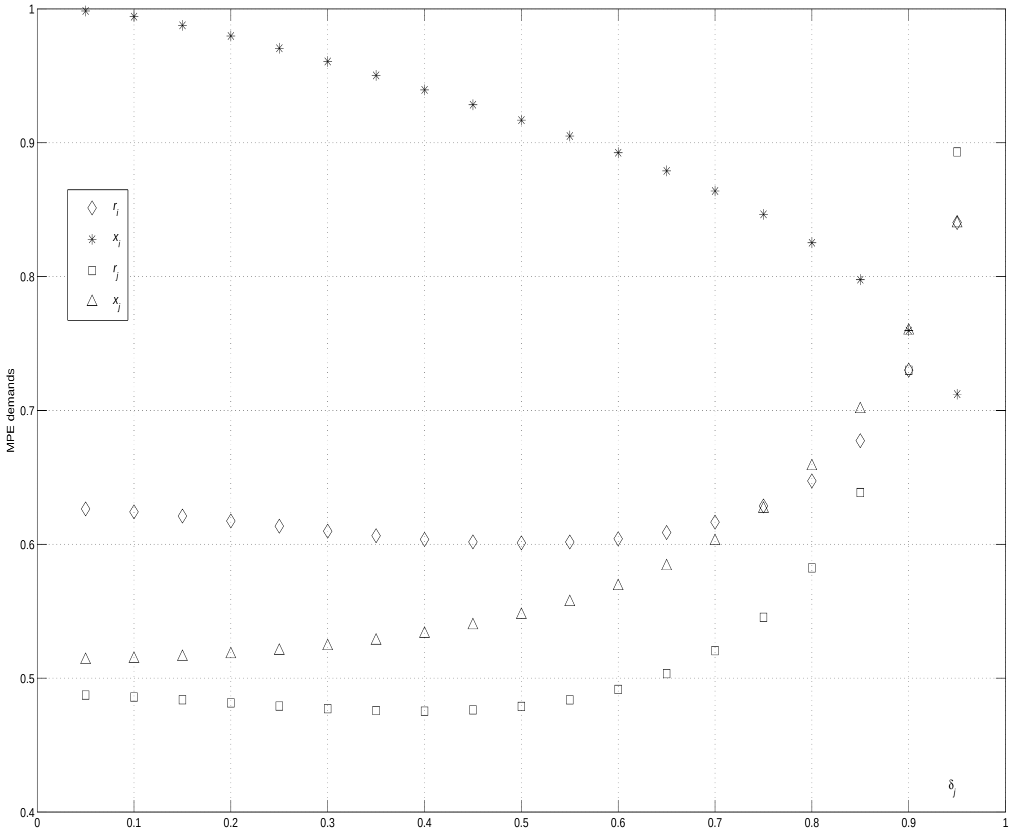

Figure 3. MPE demands for $\eta = 1/2$, $l = 1.1$ and $(\alpha_i, \delta_i) = (0.8, 0.9)$.

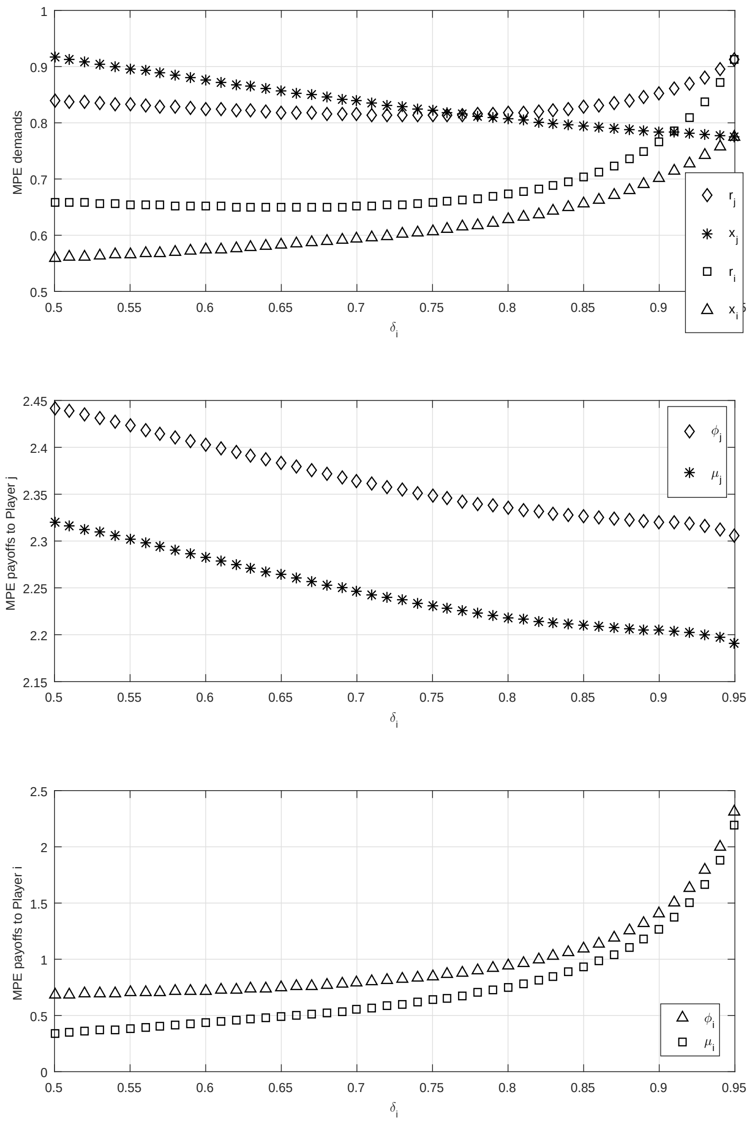

Figure 4a. MPE for $\eta = 2/3$, $l = 1.2$ and $(\alpha_j, \delta_j) = (0.8, 0.95)$.

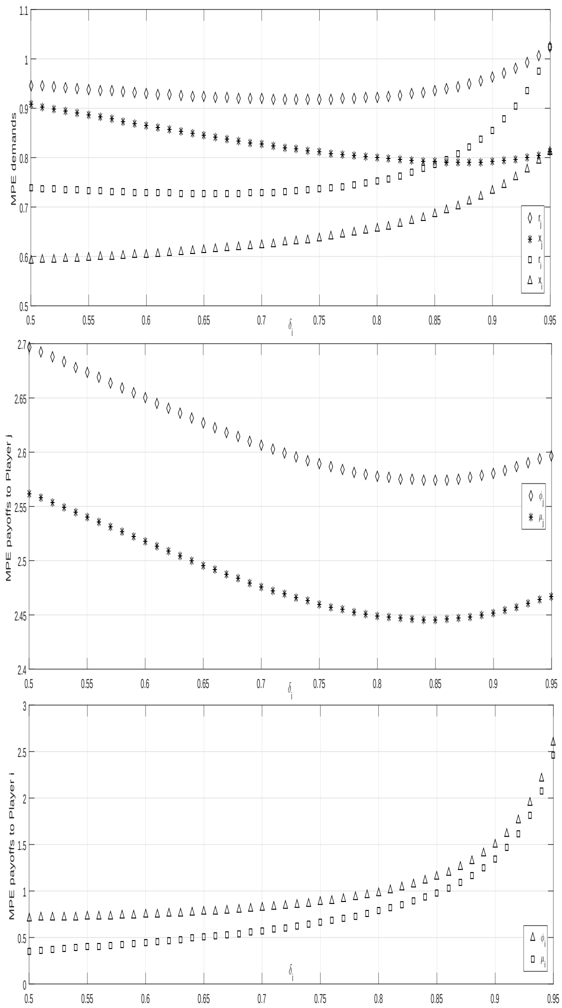

Figure 4b. MPE for $\eta = 2/3$, $l = 1.3$ and $(\alpha_j, \delta_j) = (0.8, 0.95)$.

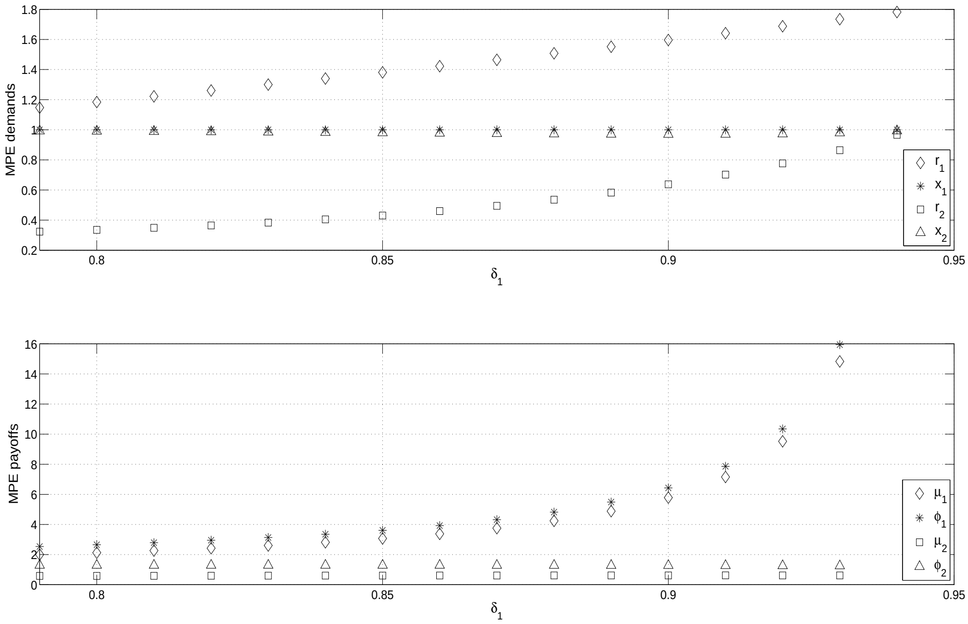

Figure 5. MPE for η=4/5, l=1.8, (α_2, δ_2)=(0.44,0.45).

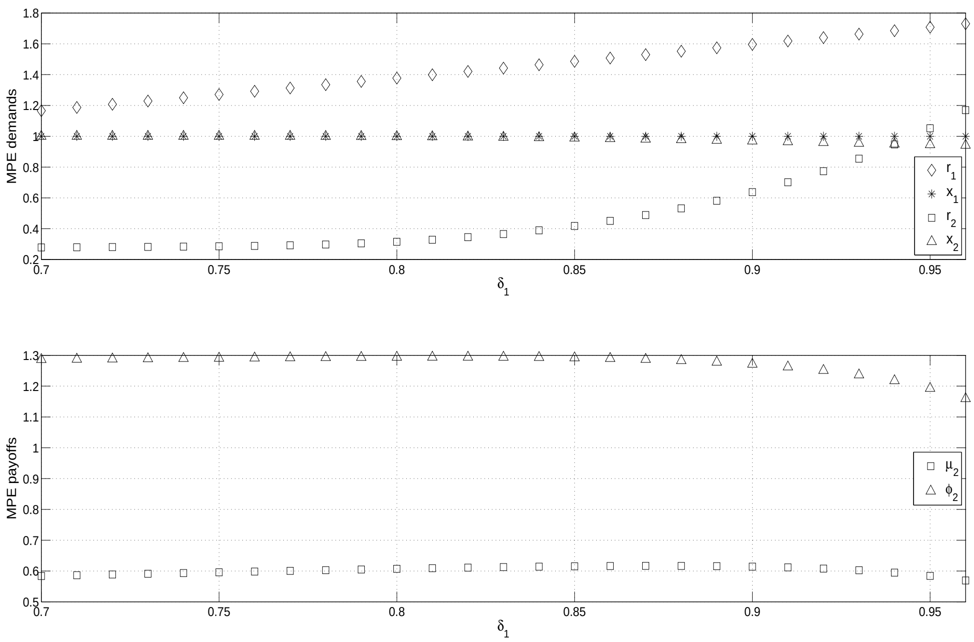

Figure 6. MPE for η=4/5, l=1.8 (α_2, δ_2)=(0.44,0.45), α_1 =0.8973.

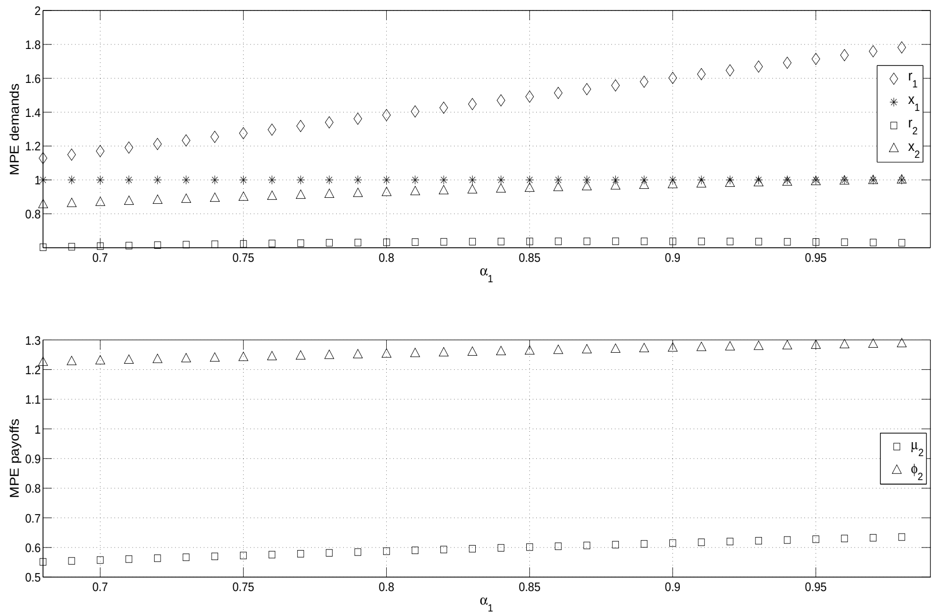

Figure 7. MPE for η=4/5, l=1.8, (α_2, δ_2)=(0.44,0.45), δ_1=0.9.