A closer look at the relationship between life expectancy and economic growth*

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Abstract

We first provide a nonparametric inference of the relationship between life expectancy and economic growth on an historical data for 18 countries over the period 1820-2005. The obtained shape shows up convexity for low enough values of life expectancy and concavity for large enough values. We then study this relationship on a benchmark model combining "perpetual youth" and learning-by-investing. In such a benchmark, the generated relationship between life expectancy and economic growth is shown to be strictly increasing and concave. We finally examine a model departing from "perpetual youth" by assuming age-dependent survival probabilities. We show that life-cycle behavior combined with age-dependent survival laws can reproduce our empirical finding.

Keywords: Life expectancy, economic growth, perpetual youth, age-dependent mortality, nonparametric estimation

JEL classification: O41, I20, J10

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1 Introduction

Increase in life expectancy is often associated with higher economic growth. A 1998 World Bank study showed that life expectancy displays a strong tendency to improve with per capita income, ranging from as low as 37 years in Sierra Leone to as high as 77 in Costa Rica, more than 12 times richer. Bhargava et al. (2001) used a parametric panel data specification and found that the dynamics of demography indicators such as lagged life expectancy variable is a significant predictor of economic growth. Chakraborty (2004) developed a theoretical model and checked its empirical consistency using a parametric cross-country regression. The author found that life expectancy has a strong and positive effect on capital accumulation.

Yet the insightful work of Kelley and Schmidt (1994, 1995) also clearly highlighted that the relationship between economic growth and longevity is far from linear. In their celebrated 1995 paper, they examined the economic-demographic correlations within parametric panel data framework (89 countries over three decades: 1960-1970, 1970-1980, 1980-1990). As in Brander and Dowrick (1994) and Barlow (1994), the authors also attempted to explicitly incorporate the dynamics of demographic effects by including both contemporaneous and lagged effects of crude birth and death rates. They principally found that demographic processes matter considerably in economic development but in a complex way. Indeed, for the 30-year panel, they observed that population growth has a negative impact on economic growth. Moreover, an increase in the crude birth rate reduces economic growth (eventually through the channel of a negative dependency rate on saving), while a decrease in the crude death rate increases economic growth. For the latter, it seems that in less developed countries, mortality reduction is clustered in the younger and/or working ages. In contrast, in the developed countries, such gains occurred in the retired cohort. Kelley and Schmidt (1995) concluded that population growth is not all good or all bad for economic growth: both elements coexist.

Since the publication of the highly influential paper of Kelley and Schmidt (1995), the relationship between demographic variables and economic development has been the subject of plenty of papers in the economic growth and economic demography literatures. In particular, Boucekkine et al. (2002, 2003, 2004), Boucekkine et al. (2007) have already built and tested some models which effectively deliver the same message: the relationship between economic development and longevity is nonlinear and essentially non-monotonic. All these models are based on a single growth engine, human capital accumulation. The associated mechanisms is the following: (i) a higher life expectancy is likely to lengthen the schooling time, thus inducing a better education and better conditions for economic development; (ii) but at the same time, the fraction of people who did their schooling a long time ago will rise, implying a negative effect on growth, which may be even worse if we account for voluntary retirement. Overall, the effect of increasing longevity on growth is ambiguous, and much less simple than the common view. Another paper taking the human capital accumulation approach is Kalemli-Ozcan et al. (2000), who reached similar conclusions in a much more stylized models.

In this paper, we provide a further and closer empirical and theoretical analysis of the relationship between life expectancy and economic growth relying on an historical panel data with long time series. More precisely, we use an historical panel data for 18 countries spanning over 1820-2005. We believe that such an historical panel is particularly interesting to capture the relationship between life expectancy and economic development (GDP). In particular, the data include some historical periods with very low values for both life expectancy and economic growth, which may entail a period-specific kind of relationship between the two variables.

In order to have the most flexible and neutral statistical framework, we use a nonparametric approach where no a priori parametric functional form is assumed. Most empirical studies in the literature are generally based on *ad hoc* parametric specifications with little attention paid to model robustness; yet different parametric specifications can lead to significantly different conclusions, and a functional misspecification problem is likely to occur. The main result of our work is to uncover a new kind of nonlinearity in the relationship between life expectancy and economic growth. In particular, while the economic growth rate is found to be increasing in life expectancy, this relationship is strictly convex for low values of life expectancy, and concave for high values of this variable.

Such findings cannot be reproduced within human vintage capital models of the Boucekkine *et al.* type. In such models, the obtained relationship is typically hump-shaped under certain conditions. We therefore propose an alternative theory which captures much more naturally the convex-concave nature of the relationship between longevity and economic growth. To this end, we move to simpler models with physical capital accumulation. We study how life-cycle behaviour combined with a physical capital accumulation engine yielding endogenous growth as in Romer's learning-by-investing (1986) can generate the convex-concavce shape. We neatly show that the outcome relies on the demographic structure assumed. Under perpetual youth, like in Blanchard (1985), such a relationship cannot be generated. However, more realistic demographics, and precisely more realistic survival laws, can do the job. We prove this by combining the survival law of Boucekkine et al. (2002) and Romer's learning-by-investing. The intuitions behind the results will be clear along the way. In particular, the assumption of age-dependent mortality rates taken in the more realistic modeling is absolutely crucial. Because people with different ages have different lifetimes in such a case, they will have different *effective* planning horizons, and notably different saving decisions. This will be shown to crucially matter in the shape of the relationship between longevity and economic development. Previous contributions merging Blanchard-like structures and physical capital accumulation can be found in Aisa and Pueyo (2004) and Echevarria (2004). None uses the realistic demographic modeling considered in this paper.

The paper is organized as follows. Section 2 presents the empirical framework and results. Section 3 considers the benchmark model merging Blanchard and Romer structures. Section 4 introduces the model with realistic demographics, and Section 5 studies the associated aggregation formulas, including a comparison with the benchmark case. Section 6 examines the properties of the resulting balanced growth paths, and finally Section 7 establishes both analytically and numerically the convex-concave shape under realistic demographics. Section 8 concludes.

2 An empirical inspection using historical data

The complex dependence of life expectancy on income till a certain threshold where the shape can reverse suggests to model empirically the growth rate of GDP using a flexible nonparametric framework. Furthermore, in our empirical setting, we follow the bulk of the literature but we do not control for all possible determinants for the growth rate of GDP. Seeveral arguments can be put forward in support of our choice. The first, obvious one, concerns historical data limitations. In this respect, it is important to note that using panel methods that sweeping country effects away allows us to control implicitly for any time invariant determinant. The second obvious and more important point is that, we are not concerned here with obtaining the best predictions for the growth rate of GDP but with the *shape* of the relationship between the latter and life expectancy. In this respect, determinants of the growth rate of GDP which are not correlated with life expectancy become irrelevant. Moreover the impact of determinants which *are* correlated with life expectancy will be captured via life expectancy. Depending on the question asked, this can be seen as a drawback or as an advantage. It is a drawback if we purport to determine the ceteris paribus impact of life expectancy on the growth rate of GDP – but what list of regressors would guarantee this? It is an advantage if we are interested in the global effect of life expectancy, including indirect effects linked with omitted variables.

2.1 The statistical specification

We use a Generalized Additive Model (hereafter GAM) for panel data.[1] Additive models are widely used in theoretical economics and statistics. Deaton and Muellbauer (1980) provides examples in which a separable structure is well designed for analysis and important for interpretability. From statistical viewpoint, the GAM specification has the advantage of avoiding the 'curse of dimensionality' which appears in nonparametric regressions when many explanatory variables are accounted for. It also allows to capture non-linearities and heterogeneity in the effect of explanatory variables on the response variable. Moreover, the statistical properties (optimal rate of convergence and asymptotic distribution) of the estimator is well known (see e.g., Stone, 1980). The structure of the model is given by

$$y_{it} = \sum_{j=1}^{p} f_j(\mathbf{x}_{it}^j) + \mu_i + \varepsilon_{it}, \qquad i = 1, \cdots, N; \quad t = 1, \cdots, T, \tag{1}$$

[1]See e.g. Hastie and Tibshirani (1990) and Stone (1985) for further details on GAM.

where y_{it} denotes the response variable (here the growth rate of GDP per capita), $\mathbf{x}_{it}^j$s are j explanatory variables for $j = 1, \cdots, p$ (here $\mathbf{x}$ denotes the life expectancy at birth), the f_j are unknown univariate functions to be estimated, μ_i is unobserved individual specific effects for which we allow arbitrary correlation with $\mathbf{x}_{it}^j$. Thus, we make no assumption on $\mathbb{E}(\mu_i|\mathbf{x}_{it}^j)$ for any set of dates $t = 1, \cdots, T$. We assume that errors ε_{it} are independent and identically distributed, but no restriction is placed on the temporal variance structure. Relation (1) is a fixed effect GAM. The unobserved effect μ_i can be eliminated by differentiating or computing the within transformation. Lagging relation (1) by one period and subtracting yields

$$y_{it} - y_{i,t-1} = \sum_{j=1}^{p} f_j(\mathbf{x}_{it}^j) - \sum_{j=1}^{p} f_j(\mathbf{x}_{i,t-1}^j) + \eta_{it}, \tag{2}$$

where $\eta_{it} = \varepsilon_{it} - \varepsilon_{i,t-1}$, and we assume (*first difference* assumption, FDA) that $\mathbb{E}(\eta_{it}|\mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j) = 0$, for $i = 1, \cdots, N$ and $t = 2, \cdots, T$. It should be noticed that the latter assumption is weaker than that of strict exogeneity which drives the within estimator (see, e.g., Wooldridge, 2002).[2] The FDA assumption identifies the functions

$$\mathbb{E}\left[y_{it} - y_{i,t-1}|\mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j\right] = \sum_{j=1}^{p} f_j(\mathbf{x}_{it}^j) - \sum_{j=1}^{p} f_j(\mathbf{x}_{i,t-1}^j), \tag{3}$$

with the norming condition $\mathbb{E}[f_j(\mathbf{x}_{it}^j, \mathbf{x}_{i,t-1}^j)] = 0$, since otherwise there will be free constants in each of the functions. We base our estimation on the 'backfitting algorithm' (Hastie and Tibshirani, 1990). For a given j, let us denote $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$ the estimates of $f(\mathbf{x}_{it})$ and $f(\mathbf{x}_{i,t-1})$ respectively. Then, a more precise estimator[3], say $\hat{\hat{f}}$, can be obtained as a weighted average of $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$:

$$\hat{\hat{f}} = \frac{1}{2}\left[\hat{f}(\mathbf{x}_{it}) + \hat{f}(\mathbf{x}_{i,t-1})\right] \tag{4}$$

Below, we apply this methodology to estimate the interplay between the growth rate of GDP per capita and life expectancy at birth.

2.2 Data

We use historical panel data for 18 countries spanning over 1820-2005.[4] As already mentioned, the variables under investigation are the growth rate of GDP per capita and life expectancy at birth. Data on GDP per capita have been collected from 'The World Economy: Historical Statistics OECD Development Centre'. We use GDP per capita at 1990 International Geary-Khamis dollars. Life expectancy data are collected from The Human Mortality Database (University of

[2] Here, strict exogeneity precludes any feedback from the current value of the growth rate of GDP per capita on future values of life expectancy.

[3] This is particularly useful in case where the shape of the two estimates are closely related.

[4] Australia, Austria, Belgium, Canada, Denmark, Finland, France, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, USA.

California, Berkeley, and Max Planck Institute for Demographic Research).[5] Life expectancy at birth is the number of years that a newborn baby is expected to live if the age-specific mortality rates effective at the year of birth apply throughout his or her lifetime.

The density estimates of the variables of interest show a unimodal distribution for the growth rate of GDP per capita (about 2.5%) and a bimodal distribution for life expectancy at birth (about 45 years age for the first mode and 74 years age for the second).[6] For the latter, the second mode clearly dominates the first. As a result, we can argue that our sample contains both an important proportion of countries with working age and retired people.

2.3 Estimation results

As mentioned above, our estimates $\hat{f}(\mathbf{x}_{it})$ and $\hat{f}(\mathbf{x}_{i,t-1})$ have closely related shape. We then plot in Figure 1 the weighted average $\hat{\hat{f}}$. With respect to the relational structure, a study of the graph gives the first hand impression that life expectancy effect on per capita income growth rate is highly non-linear. To test for the significance of non-linearity in the statistical specification, we use the 'gain' statistic (see, Hastie and Tibshirani, 1990 for details).[7] The 'gain' is computed as $178.014 > \chi^2(24.008) = 36.424$ at the 5% level. As a result, there is a strong evidence of non-linearity.

Insert Figures 1 and 2

This finding provides a new evidence, in contrast to the linearity assumption of the wide array of empirical models of the demography-economic growth relation built on parametric framework. The curvature suggests that the relation between economic growth and life expectancy involves far more complex mechanism. In the linear case, demographic shocks may eventually wither away with little or no long run effect on economic growth, whereas non-linearity can induce the shocks to work in a much more intricate way.

Our empirical specification is flexible enough to account for the complex way life expectancy does affect economic growth. The main lesson which emerges from Figure 1 is that the relationship between life expectancy and GDP growth rate, while roughly increasing, has quite varying concavity depending on the value of life expectancy. The relationship is for example convex for low life expectancy values, and concave for large enough values of this variable. The

[5] `http://www.mortality.org/`

[6] The plots of the distributions are not reported. Densities are estimated nonparametrically using the kernel method. Results of these estimations are available upon request.

[7] Intuitively, the 'gain' is the difference in normalized deviance between the GAM and the parametric linear model. A large 'gain' indicates a lot of non-linearity, at least as regards statistical significance. The distribution of this statistic is approximated by a chi-square $\chi^2\,(df = df_g - df_l)$, where df_g denotes the degree of freedom of the GAM. It is computed as the trace of $2\mathbf{S} - \mathbf{S}\mathbf{S}'$ where $\mathbf{S}$ is the smoothing matrix, and df_l is the degree of freedom of the parametric linear model. Here we use the first difference linear model $y_{it} - y_{i,t-1} = \beta(\mathbf{x}_{it} - \mathbf{x}_{i,t-1}) + \varepsilon_{it} - \varepsilon_{i,t-1}$, which is then estimated by ordinary least squares. In that case, $\mathbf{S}$ turns out to be the matrix of orthogonal projection: $\mathbf{S} = \mathbf{Z}\,(\mathbf{Z}'\mathbf{Z})^{-1}\,\mathbf{Z}'$, where $\mathbf{Z}$ denotes the matrix of regressors which does stack up elements of $\mathbf{x}_{it} - \mathbf{x}_{i,t-1}$.

finding has been found robust to two modifications. Lagging life expectancy by one period as in Barghava et al (2001) will not affect the convex-concave shape. Moreover, running the same non-parametric estimation on averaged variables over successive 20-years long periods does not smooth out the shape. Hereafter, we study to which extent life-cycle behavior under different demographic structures within an endogenous growth set-up can explain this shape.

3 The benchmark: Blanchard meets Romer

We first very briefly display the basic structure of Blanchard-like models. More details can be found in Blanchard (1985). We then introduce learning-by-investing as in Romer (1986).

3.1 Demography

We assume that at every instant, a cohort is born with constant size π. Each member of any cohort has a constant instantaneous (flow) probability to die equal to p. Therefore, an agent born at μ (generation, cohort or vintage μ) has a probability $e^{-p(z-\mu)}$ to survive at $z > \mu$ and life expectancy is constant and equal to $1/p$. At time z, the size of a cohort born at time μ, say $T(\mu, z)$ is equal to $\pi e^{-p(z-\mu)}$. In order for the size of total population to be normalized to 1, that is $\int_{-\infty}^{z} T(\mu, z)d\mu = 1$, we require $\pi = p$.

3.2 The consumer's problem

We assume that the utility function is of the standard log form. The intertemporal utility is then:

$$\int_{\mu}^{\infty} \ln C(\mu, z)e^{-(p+\rho)(z-\mu)}dz \tag{5}$$

where $C(\mu, z)$ denotes the consumption of an individual belonging to generation μ at time z, and ρ is the intertemporal discount rate of the utility. Each individual holds an amount of wealth $R(\mu, z)$ which is equal to the accumulated excess of non interest earnings over consumption outlays, plus accumulated interest charges at time z. Agents are constrained to maintain a positive wealth position, and have no bequest motive. Since individual age is directly observed, the annuity rate of interest faced by an individual of age $z-\mu$ is the sum of the world interest rate and his instantaneous probability of dying $(r(z) + p)R(z - \mu)$ at each time, as a payment from the insurance companies if he is still alive. After his death, all his wealth goes to the insurance firms. We consider that all individuals supply entirely their available time, normalized to one, and are paid at the wage rate, $\omega(z)$, for every date z. Under these considerations the budgetary constraint is:

$$\dot{R}(\mu, z) = (r(z) + p)R(\mu, z) + \omega(z) - C(\mu, z) \tag{6}$$

with

$$\lim_{z\to\infty} R(\mu, z) \; e^{-\int_s^z [r(x)+p]dx} = 0$$

The consumer maximizes Equation (5) subject to (6) with the initial condition $R(\mu,\mu)$ given. We shall assume for simplicity that $R(\mu,\mu)=0$ for every μ. The associated Hamiltonian is

$$H = \ln C(\mu,z)e^{-(p+\rho)(z-\mu)} + \lambda\left[(r(z)+p)R(\mu,z)+\omega(z)-C(\mu,z)\right]$$

where λ is the co-state variable associated with the state variable $R(.)$. The resulting first order conditions are:

$$\begin{aligned}
\frac{\partial H}{\partial C(\mu,z)} &= 0 \Leftrightarrow \frac{1}{C(\mu,z)}e^{-(p+\rho)(z-\mu)} = \lambda \\
\frac{\partial H}{\partial R(\mu,z)} &= -\dot{\lambda} = \lambda\left[r(z)+p\right] \Rightarrow \frac{\dot{\lambda}}{\lambda} = -\left[r(z)+p\right] \\
\frac{\partial H}{\partial \lambda} &= \dot{R} = \left[r(z)+p\right]R(z,\mu)+\omega(z)-C(z,\mu)
\end{aligned}$$

Using the equations above, the optimal consumption is such that:

$$\frac{\partial C(\mu,z)/\partial z}{C(\mu,z)} = \frac{\dot{C}(\mu,z)}{C(\mu,z)} = r(z)-\rho \tag{7}$$

The relation (7) is the traditional Euler equation describing optimal consumption behavior over time. It shows that the optimal path of consumption is closely determined by the difference between the interest rate and the pure rate of time preference. The proposition below describes the relation between the consumption and the total wealth of the individual.

Proposition 1

$$C(\mu,z) = (p+\rho)[R(\mu,z)+D(\mu,z)] \tag{8}$$

where

$$D(\mu,z) = \int_z^\infty \omega(v)e^{-\int_z^v [r(s)+p]ds}dv$$

$D(\mu,z)$ is the human wealth at z of an individual born at μ. It's the present value of the future stream of labor income. $D(\mu,z)$ is obtained by integrating forward the individual's dynamic budget constraint (Equation 6).

3.3 Aggregation

As in the standard Blanchard case, for any vintage $x_{\mu,z}$, the aggregate magnitude $X(z)$ is

$$X(z) = \int_{-\infty}^z x(\mu,z)T(\mu,z)d\mu$$

where $T(\mu,z) = pe^{-p(z-\mu)}$ is the size of the cohort μ. The following aggregation formulas have been proved by Blanchard (1985, pp. 228–230). Their simplicity explains to a large extent the popularity of the model.

Proposition 2 *The evolution of the aggregate consumption is:*

$$\dot{C}(z) = (r - \rho)C(z) - p(p + \rho)R(z) \tag{9}$$

where $R(z)$ is the aggregate nonhuman wealth. The law of motion of aggregate human wealth is given by

$$\dot{D}(z) = [r + p]D(z) - w(z)$$

while the law of motion of aggregate nonhuman wealth follows

$$\dot{R}(z) = rR(z) + w(z) - C(z)$$

3.4 The firm's problem

As in Blanchard (1985, p. 232), we consider a closed economy. Nonhuman wealth, $R(t)$, is then equal to the stock of capital $K(t)$, and the interest rate is equal to the marginal productivity of capital. In such a case, either the wage and the interest rate are determined by the usual neoclassical conditions depending on the pace of capital accumulation. In this paper, we depart from the original Blanchard's paper by incorporating an endogenous growth engine, that is the learning-by-doing devise of Romer (1986). Firms accumulate capital, and the more they accumulate machines, the more they become expert in them, which boosts their productivity. Productivity growth, and ultimately economic growth, is therefore a side-product of capital accumulation in this model. To keep things as simple as possible, we hereby describe briefly the firm model (a more detailed exposition can be found in Romer, 1986). The production function of a representative firm i is:

$$Y_i = B(K_i)^{\varepsilon}(AL_i)^{1-\varepsilon}, \qquad 0 < \varepsilon < 1$$

where ε denotes the capital share. We suppose that there are N identical and perfectly competitive firms. K_i and L_i are respectively capital and labor factors of firm i. A is labor-saving technical progress. Note that A is not indexed by i, it represents the stock of knowledge of the whole economy, and such a stock is supposed to be outside the control of any particular firm: it is not appropriable. Let us now come back to Romer. A is an increasing function of the capital stock accumulated by all the firms of the economy (note this is consistent with the no-appropriability specification outlined above). As in Romer, we simply assume that A is proportional to the aggregate capital stock $K = \sum_1^N K_i = NK_i$ since firms are identical. Thus the function of production of the firm i becomes:

$$Y_i = B(K_i)^{\varepsilon}(KL_i)^{1-\varepsilon} \tag{10}$$

where K is of course out of the control of any firm i under the traditional assumptions of perfect competition (notably under N large enough). Under zero capital depreciation, the profit function of the firm i is $\pi_i = Y_i - wL_i - rK_i$, and the maximization of this profit with respect to K_i and L_i give the traditional conditions:

$$\frac{\partial \pi_i}{\partial K_i} = \varepsilon B \left(\frac{K}{N}\right)^{\varepsilon-1} \left(K\frac{L}{N}\right)^{1-\varepsilon} - r = 0 \quad (11)$$

$$\frac{\partial \pi_i}{\partial L_i} = (1-\varepsilon) B \left(\frac{K}{N}\right)^{\varepsilon} K^{1-\varepsilon} \left(\frac{L}{N}\right)^{-\varepsilon} - w = 0 \quad (12)$$

which respectively yield $r = \varepsilon B L^{1-\varepsilon}$ and $w = (1-\varepsilon) BKL^{-\varepsilon}$.

3.5 General equilibrium

As usual in one-sector growth models driven by capital accumulation, we are able to summarize the dynamics of the model at general equilibrium in two equations depending on aggregate consumption and physical capital.[8] As just mentioned above, nonhuman wealth is equal to $K(z)$ at general equilibrium since we are in a closed economy. Equation (9) can therefore be rewritten as

$$\dot{C}(z) = (r-\rho)C(z) - p(p+\rho)K(z) \quad (13)$$

with $r = \varepsilon B$ since the size of (active) population is normalized to one by construction. As usual, the second equation is provided by the resource constraint of the economy, which could be straightforwardly transformed at equilibrium as the law of motion of physical capital, that is:

$$\begin{aligned} \dot{K} &= Y - C(z) \\ &= NB\left(\frac{K}{N}\right)^{\varepsilon} K^{1-\varepsilon}\left(\frac{L}{N}\right)^{1-\varepsilon} - C(z) \end{aligned}$$

which yields (with the normalization $L = 1$)

$$\dot{K} = BK - C(z) \quad (14)$$

We know study the existence of balanced growth solutions to the system (13)-(14)

3.6 Balanced growth paths and the relationship between longevity and growth

We shall first define steady state growth paths (or balanced growth paths). Such paths occur when aggregate consumption, capital stock (or investment) and thus output grow at a constant rate, say g. In more mathematical terms, we are seeking for exponential solutions with constant exponents g. Because of the resource constraint of the economy, that is: $Y(z) = C(z) + \dot{K}(z)$, the three variables should grow at the same exponential rate if we want the ratios $\frac{C}{Y}$ and $\frac{I}{Y}$ to be constant along the steady state growth paths, which is a required economic regularity. As usual in endogenous growth theory, we have indeterminacy in the long-run level of the variables as the two-dimensional system (13)-(14) cannot allow to compute the two long-run levels (of

[8] Plus the traditional boundary conditions.

consumption and capital respectively) plus the unknown growth rate g. We shall proceed here by the traditional dimension reduction method. Precisely, we focus on the two variables, ratio consumption to capital, $\frac{C}{K} = X$, and the growth rate $g = \frac{\dot{C}(z)}{C(z)} = \frac{\dot{K}(z)}{K(z)}$, and rewrite the system (13)-(14) at the balanced growth path in terms of these two variables. We get:

$$\begin{aligned} \frac{\dot{C}(z)}{C(z)} &= g = (r-\rho) - p(p+\rho)X^{-1} \\ \Longrightarrow X &= \frac{p(p+\rho)}{r-\rho-g} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \frac{\dot{K}}{K} &= B - \frac{\bar{C}}{\bar{K}} \\ g &= B - X \end{aligned} \tag{16}$$

If $p = 0$, we recover the traditional demographic structure in growth theory, and the counterpart outcomes. Let us depart from this case and assume $p \neq 0$. By combining the Equations (15) and (16), it turns our that the in this case long-run growth rate g should solve a second-order polynomial as in Aisa and Pueyo (2004):

$$-g^2 + g(B + r - \rho) + B(\rho - r) + p(p + \rho) = 0 \tag{17}$$

In contrast to Aisa and Pueyo, we are able to analytically characterize the associated properties.

Proposition 3 *Provided $B(r-\rho) > p(p+\rho)$, the model displays two strictly positive values for the long-run growth rate g. However, only a single value, the lower, is compatible with a positive ratio consumption to capital.*

Proof. The discriminant of the second-order g-equation (17) is:

$$\Delta = (B + r - \rho)^2 + 4\left(B(\rho - r) + p(p + \rho)\right)$$

which can be trivially rewritten as:

$$\Delta = (B - r + \rho)^2 + 4p(p + \rho)$$

Therefore, since $p > 0$, we always have two distinct roots. The largest root is necessarily strictly positive once $r > \rho$, which is ensured under the sufficient condition $B(r-\rho) > p(p+\rho)$. Call it g^M:

$$g^M = \frac{B + r - \rho + \sqrt{\Delta}}{2}$$

The second root, say g^m, with $g^m = \frac{B+r-\rho-\sqrt{\Delta}}{2}$, has the sign of:

$$(B + r - \rho - \sqrt{\Delta})(B + r - \rho + \sqrt{\Delta}) = 4\ \left(B(r - \rho) - p(p + \rho)\right)$$

which is positive under $B(r-\rho) > p(p+\rho)$. This proves the first part of the proposition.

To prove the second part, notice that g^M is necessarily bigger than B:

$$2g^M = B + r - \rho + \sqrt{\Delta} \geq B + r - \rho + (B - r + \rho) = 2B$$

since

$$\sqrt{\Delta} = \sqrt{(B - r + \rho)^2 + 4p(p + \rho)} \geq B - r + \rho$$

and

$$B - r + \rho = (1 - \epsilon)B + \rho > 0$$

Because the ratio consumption to capital is determined by $X = \frac{p(p+\rho)}{r-\rho-g}$, X is necessarily negative if $g \geq B$. Thus, g^M should be ruled out.

In contrast, g^m checks the inequality $g^m < r - \rho$, which guarantees the positivity of the ratio X. Indeed:

$$2g^m = B + r - \rho - \sqrt{\Delta} \leq B + r - \rho - (B - r + \rho) = 2(r - \rho)$$

which ends the proof of the proposition. ■

Two comments are in order. First of all, the apparent multiplicity that comes from the second-order polynomial equation is actually fictitious. Therefore, contrary to what is suggested in Aisa and Pueyo (2004), and though we effectively have two distinct and strictly positive values for g, the largest value is simply incompatible with the positivity of the ratio consumption to capital. Second, our sufficient condition is actually not restrictive at all. In particular, if we have in mind that p and ρ are small number compared to the productivity parameter B, the inequality $B(r - \rho) > p(p + \rho)$ should hold. In other words, our proposition is in particular valid as long as the economy is productive enough and the mortality rate p is small enough. Otherwise, a combination of low productivity (low B) and high mortality (large p) can induce negative growth, which is actually reflected in Figures 1 and 2.[9] In this sense, our model behaves very well. The relationship between growth and longevity induced by the model is even neater as summarized in the following proposition:

Proposition 4 *Under the assumption of Proposition 3, the unique admissible long-run growth rate of the economy is a strictly increasing, strictly concave function of longevity. In other words,* g^m *is a strictly decreasing, strictly convex function of* p.

The proof is trivial by simple differentiation of $g^m = \frac{B+r-\rho-\sqrt{\Delta}}{2}$ and its first derivative with respect to p, given $\Delta = (B - r + \rho)^2 + 4p(p + \rho)$. The main conclusion of this section is therefore that the benchmark model obtained by combination of the perpetual youth model of

[9]We shall not however consider balanced growth paths with negative growth rates, to make things simple.

Blanchard (1985) and the learning-by-investing engine of Romer (1986) delivers a quite simple picture of the relationship between longevity and growth: the relationship is strictly monotonic and strictly concave, and it does not exhibit any first-order difference between the case of low and high life expectancy countries. We argue that this property strongly relies on the perpetual youth assumption, that is on the fact that survival probabilities are age-independent. We relax this assumption hereafter.

4 The model with realistic demography

Rather than a typical Blanchard-like set-up, we choose the survival law previously put forward by Boucekkine et al. (2002). The probability of surviving until age a ($a = z - \mu$) for any individual of cohort μ is

$$m(a, \mu) = \frac{e^{-\beta(\mu)a} - \alpha(\mu)}{1 - \alpha(\mu)} \tag{18}$$

and the probability of death at age a is

$$F(a, \mu) = 1 - \frac{e^{-\beta(\mu)a} - \alpha(\mu)}{1 - \alpha(\mu)} = 1 - m[\alpha(\mu), \beta(\mu), (z - \mu)] = \frac{1 - e^{\beta(\mu)a}}{1 - \alpha(\mu)} \tag{19}$$

with $\beta(\mu)$ an indicator of survival for old persons, and $\alpha(\mu)$ is an indicator of survival for young persons. We suppose $\beta(\mu) < 0$, and $\alpha(\mu) > 1$ as in Boucekkine et al. (2002) in order to generate a concave survival law as observed in real life, as described in Figure 3.

Insert Figure 3

The maximum age possible for individuals of cohort μ is given by: $m(a, \mu) = 0$, that it is, $A_{\max} = -\frac{\ln(\alpha(\mu))}{\beta(\mu)}$. The expression of the instantaneous probability of dying is then:

$$S(a) = \frac{\partial F(a)/\partial z}{m} = \frac{-\partial m/\partial z}{m} = \frac{\beta(\mu)e^{-\beta(\mu,z-\mu)}}{e^{-\beta(\mu,z-\mu)} - \alpha(\mu)} \tag{20}$$

Life expectancy is:

$$E = \int_{\mu}^{\infty} (z - \mu) \frac{\beta(\mu)e^{-\beta(\mu,z-\mu)}}{1 - \alpha(\mu)} dz = \frac{\alpha(\mu) \ln \alpha(\mu)}{\beta(\mu)(1 - \alpha(\mu))} + \frac{1}{\beta(\mu)} \tag{21}$$

For $\beta(\mu) > 0$ and $\alpha(\mu) \to 0$, one finds the Blanchard result. The size of the population at time z is

$$T_z = \int_{z-A_{\max}}^{z} \xi e^{n\mu} \frac{e^{\beta(\mu,z-\mu)} - \alpha(\mu)}{1 - \alpha(\mu)} d\mu \tag{22}$$

Without loss of generality, we shall set $n = 0$ hereafter.

4.1 The model

With respect to the benchmark model, we keep the production side unchanged but we modify substantially the consumer side in order to incorporate more realistic demographics. We therefore concentrate on the latter problem hereafter. We shall consider the optimization problem of an individual of a generation μ. For ease of the exposition we may omit the dependence of the demographic parameters $\alpha(\mu)$ and $\beta(\mu)$ on μ. Assuming that the instantaneous utility derived from consumption is logarithmic for an individual born in μ still living in z, the intertemporal utility is

$$\int_{\mu}^{\mu+A_{\max}} \ln C(\mu, z) \frac{e^{-\beta(z-\mu))} - \alpha}{1-\alpha} e^{-\rho(z-\mu)} dz \tag{23}$$

One can consider that utility after the individual dies is equal to zero, then in the intertemporal utility we can replace $\mu + A_{\max}$ by ∞. On the other hand, and contrary to Boucekkine et al. (2002), we assume that there is no disutility of work. As in the precedent section we find the traditional Euler equation

$$\frac{\partial C(\mu, z)/\partial z}{C(\mu, z)} = \frac{\dot{C}}{C} = r(z) - \rho + S(\mu, z) - S(\mu, z) = r(z) - \rho \tag{24}$$

That is to say $C(\mu, z) = C(\mu, \mu) e^{\int_{\mu}^{z}(r(s)-\rho)ds}$. Consumption over time can be characterized much more finely using the approach highlighted in Faruqee (2003).

Proposition 5

$$C(\mu, z) = \phi(\mu, z)[R(\mu, z) + D(\mu, z)] \tag{25}$$

where

$$\phi(\mu, z) = \frac{1}{\int_{z}^{\infty} e^{-\int_{z}^{v}[\rho + S[\alpha,\beta,(x-\mu)]dx} dv}$$

Proof. See appendix. ■

Corollary 1 *If $S(\mu, z) = p$, then Equation (25) degenerates into the Blanchard's case: $C(\mu, z) = (p + \rho)[R(\mu, z) + D(\mu, z)]$.*

$\phi(\mu, z)$ is the marginal propensity to consume. Contrary to the Blanchard case, the marginal propensity to consume is no longer constant, it is a much more complicated and depends in particular on age and generation characteristics:

$$\phi(\mu, z) = \frac{\rho(\rho+\beta)(e^{-\beta a} - \alpha)}{\rho\left[e^{-\beta a} - e^{(\rho+\beta)(a-A_{\max})}\right] + \alpha(\rho+\beta)\left[e^{\rho(a-A_{\max})} - 1\right]} \tag{26}$$

and

$$\phi(\mu, \mu) = \frac{\rho(\rho+\beta)(1-\alpha)}{\rho\left[1 - e^{-A_{\max}(\rho+\beta)}\right] + \alpha(\rho+\beta)\left[e^{-\rho A_{\max}} - 1\right]} \tag{27}$$

One can then notice that contrary to the Blanchard case previously studied by Aisa and Pueyo (2004), the marginal propensity to consume, in addition to be age-dependent, is a definitely much more complex function of the demographic and preference parameters. In order to get a closer idea about this, let us study the evolution of $\phi(\mu,\mu)$ with respect to β, α, and ρ. We first give the exact algebraic expressions of the derivatives involved before stating a proposition and providing some numerical illustrations.

$$\frac{\partial\phi(\mu,\mu)}{\partial\beta} = \frac{e^{-(\rho+\beta)A_{\max}}\left[(1-\alpha)\left(-\rho^2+\frac{\rho^3(\rho+\beta)}{\beta}\right)\right]-\frac{e^{-\rho A_{\max}}\rho^2\alpha(1-\alpha)(\rho+\beta)^2}{\beta}+\rho^2(1-\alpha)}{\left[\rho\left(1-e^{-(\rho+\beta)A_{\max}}\right)+\alpha(\rho+\beta)(e^{-\rho A_{\max}}-1)\right]^2}$$

$$\frac{\partial\phi(\mu,\mu)}{\partial\alpha} = \frac{e^{-(\rho+\beta)A_{\max}}\rho^2(\rho+\beta)\left[\frac{(1-\alpha)\rho+\beta}{\alpha\beta}\right]-e^{-\rho A_{\max}}\rho(\rho+\beta)^2\left[1+\frac{\rho(1-\alpha)}{\beta}\right]+\beta\rho(\rho+\beta)}{\left[\rho\left(1-e^{-(\rho+\beta)A_{\max}}\right)+\alpha(\rho+\beta)(e^{-\rho A_{\max}}-1)\right]^2}$$

$$\frac{\partial\phi(\mu,\mu)}{\partial\rho} = \frac{-e^{-(\rho+\beta)A_{\max}}\rho^2(1-\alpha)\left[(\rho+\beta)A_{\max}+1\right]+e^{-\rho A_{\max}}\alpha(1-\alpha)(\rho+\beta)^2\left[\rho A_{\max}+1\right]}{\left[\rho\left(1-e^{-(\rho+\beta)A_{\max}}\right)+\alpha(\rho+\beta)(e^{-\rho A_{\max}}-1)\right]^2} \\ \frac{+(1-\alpha)\left[-\alpha(\rho+\beta)^2+\rho^2\right]}{\left[\rho\left(1-e^{-(\rho+\beta)A_{\max}}\right)+\alpha(\rho+\beta)(e^{-\rho A_{\max}}-1)\right]^2}$$

We can then state the following property:

Proposition 6 *If the maximum age $A_{\max}$ is large enough (if α large enough and/or β close to zero), then* $\frac{\partial\phi(\mu,\mu)}{\partial\alpha}<0$, $\frac{\partial\phi(\mu,\mu)}{\partial\beta}<0$ *and* $\frac{\partial\phi(\mu,\mu)}{\partial\rho}>0$

Proof. See appendix. ■

4.2 Numerical exercises

Proposition 6 is illustrated in Figures 4, 5 and 6, where the saving rate, $s(\mu,\mu)$, which is equal to $1-\phi(\mu,\mu)$, is represented as a function of the three parameters considered. Economically speaking, the derivatives are correctly signed in Proposition 6. If demographic conditions move in such a way that life expectancy and/or the maximal age go up (α and β increasing) then the consumer will face higher horizons and save more in marginal terms. In contrast, when the impatience rate is raised (ρ growing), the propensity to consume increases, and the saving rate goes down. This is clearly reflected in Figures 4 to 6 for some reasonable parameterizations of the model.

Insert Figures 4, 5 and 6

In order to study whether the properties outlined just above are truly sensitive to the sufficient condition of Proposition 6, that is to a sufficiently large value of the maximal age, we have run more numerical experiments. A sample is given in Figures 7, 8 and 9 in which the range of values taken by the maximal age, $A_{\max}$, is much tighter than in the first case. Again, we recover the same patterns, suggesting that the properties are indeed much less fragile than what could be inferred from the statement of Proposition 6.

Insert Figures 7, 8 and 9

We end this section by considering a very important property of the model already mentioned in the introduction. In contrast to Aisa and Pueyo (2004), the saving rates or propensities to save do depend on the age of the individuals. Intuitively, the older should have the lower propensities to save. Figure 10 shows an illustration of this property of the model: as one can see, the evolution of the saving rate for an individual born at μ still living at z is clearly declining: an individual saves definitely much less when old compared to her youth. This should induce some strong non-linearity between longevity and development. If the latter relies on accumulation of physical capital, and if such an accumulation is only possible thanks to domestic savings, then a larger longevity has also a negative effect on growth by increasing the proportion of people with relatively small saving rates. Of course it is not clear at all whether this negative effect will dominate the direct positive effects of increasing longevity, but we can already argue that the relationship between the latter and economic development cannot be as simple as in the typical Blanchard-like models with physical capital accumulation.

Insert Figure 10

5 Aggregation

We start by constructing the aggregate magnitudes related to consumers behaviour as in the original work of Blanchard (1985). Regarding this particular issue, our work mimics the one of Faruqee (2003). Second, we show how Blanchard specifications can be derived from the general aggregation formulas established under age-dependent survival probabilities.

5.1 Aggregation with age-dependent survival probabilities

Before studying the growth rate of the aggregate economy, we need to define and compute some aggregate figures across generations or vintages. Given the characteristics of our model, for any vintage $x_{\mu,z}$, the aggregate magnitude $X(z)$ is computed following:

$$X(z) = \int_{-\infty}^{z} x(\mu, z)T(\mu, z)d\mu = \int_{z-A_{\max}}^{z} x(\mu, z)T(\mu, z)d\mu$$

where $T(\mu, z)$ is the size of generation μ. We start with aggregate consumption.

Proposition 7 *The aggregate consumption* $C(z) = \int_{z-A_{\max}}^{z} C(\mu, z)T(\mu, z)d\mu$ *evolves according to*

$$\dot{C}(z) = \xi\phi(z,z)D(z,z) + (r(z)-\rho)C(z) - \int_{z-A_{\max}}^{z} C(\mu,z)S(\mu,z)T(\mu,z)d\mu \tag{28}$$

with $T(\mu, z) = \xi m[\beta, \alpha, (z-\mu)]$ *and* $C(\mu, z) = C(\mu, \mu)e^{\int_{\mu}^{z}[r(s)-\rho]ds}$.

***Proof*.** From Equation (24) we can deduce

$$C(\mu, z) = C(\mu, \mu)e^{\int_{\mu}^{z}[r(s)-\rho]ds} \tag{29}$$

Aggregate consumption is

$$C(z) = \int_{-\infty}^{z} C(\mu,z)T(\mu,z)d\mu$$

Differentiating the latter equation with respect to z, one gets:

$$\dot{C} = C(z,z)T(z,z) + \int_{-\infty}^{z} \dot{C}(\mu,z)T(\mu,z)d\mu + \int_{-\infty}^{z} C(\mu,z)\dot{T}(\mu,z)d\mu$$

Using again Equation (24) we can go further:

$$\begin{aligned} \dot{C} &= C(z,z)T(z,z) + \int_{-\infty}^{z} (r(z)-\rho)C(\mu,z)T(\mu,z)d\mu + \int_{-\infty}^{z} C(\mu,z)\dot{T}(\mu,z)d\mu \\ &= C(z,z)T(z,z) + (r(z)-\rho)C(z) + \int_{-\infty}^{z} C(\mu,z)\dot{T}(\mu,z)d\mu \end{aligned}$$

We know that $T(\mu, z) = \xi m(z-\mu)$, which implies that $\dot{T}(\mu, z) = \xi \dot{m}(z-\mu)$. Since $S(z-\mu) = -\frac{\dot{m}(z-\mu)}{m}$, we deduce that $\dot{m}(z-\mu)$ is equal to $-m(z-\mu)S(z-\mu)$, and $\dot{T}(\mu, z)$ is equal to $-\xi m(z-\mu)S(z-\mu)$. As a result, the law of motion of aggregate consumption can be rewritten again as:

$$\dot{C} = C(z,z)T(z,z) + (r(z)-\rho)C(z) - \int_{-\infty}^{z} C(\mu,z)S(\mu,z)T(\mu,z)d\mu$$

$C(z, z)$ is the consumption of newly born at date z. We know by Proposition 5 that $C(z, z) = \phi(z, z)D(z, z)$ with $R(z, z) = 0$ since we assumed that a newly born agent has no financial wealth. We also have $T(z, z) = \xi m(z, z) = \xi \frac{e^{-\beta(z-z)}-\alpha}{1-\alpha}$, implying that $T(z, z) = \xi$. This yields the law of motion stated in this proposition:

$$\dot{C} = \xi\phi(z,z)D(z,z) + (r(z)-\rho)C(z) - \int_{-\infty}^{z} C(\mu,z)S(\mu,z)T(\mu,z)d\mu \quad \blacksquare$$

This law of motion is markedly different from Blanchard's (1985) as re-used by Aisa and Pueyo (2004). In our model, aggregate consumption depends on three terms: the human wealth of newly born, the Keynes-Ramsey standard term (the difference between the interest rate and the pure rate of time preference), and the expected consumption forgone by individuals dying at z. In particular this last term does not show up in the typical Blanchard aggregation formulas.

This term makes a big difference and complicates substantially the computations with respect to standard Blanchard model. The next two propositions states two useful aggregation formulas for human and non-human wealth respectively, which again show substantial departures from the standard case. The evolution of human wealth $D(z)$ below stated.

Proposition 8 *The total human wealth at z,* $D(z) = \int_{z-A_{\max}}^{z} D(\mu, z)T(\mu, z)d\mu$ *evolves according to:*

$$\begin{aligned}
\dot{D}(z) &= \xi D(z,z) - w(z)L(z) \\
&+ \int_{z-A_{\max}}^{z} \int_{z}^{z+A_{\max}} w(v)[r(z) + S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds} T(\mu, z)\, dv\, d\mu \\
&- \int_{z-A_{\max}}^{z} D(\mu, z)T(\mu, z)S(z-\mu)d\mu
\end{aligned} \tag{30}$$

***Proof*.** See appendix. ■

For the non-human wealth variable, $R(z)$, we have the following aggregation formula:

Proposition 9 *The evolution of the aggregate nonhuman wealth* $R(z) = \int_{z-A_{\max}}^{z} R(\mu, z)T(\mu, z)d\mu$ *is*

$$\dot{R}(z) = R(z,z)T(z,z) + r(z)R(z) + w(z)L(z) - C(z)$$

***Proof*.** See appendix. ■

We now turn to the computation of the steady state growth rates and their relationship with the demographic parameters.

5.2 Comparison with Blanchard

In the Blanchard's case, two basic simplifying assumptions are made:

i) *The survival and death probability are constant*

$$S(\mu, z) = p \quad \forall \mu, \forall z$$

ii) *Normalization assumptions:* $\xi = p$ *and* $T(\mu, z) = pe^{-p(z-\mu)}$*, which implies that total population is equal to 1. That means*

$$\int_{-\infty}^{z} T(\mu, z)d\mu = 1$$

These two assumptions simplify a lot the computation in that they remove two essential aspects of intergenerational heterogeneity: age-dependent survival rates and human wealth cohort-specificity. Let's have a look at the impact of these assumptions in our framework

- *The law of motion of consumption.* We have,

$$\dot{C} = \xi\phi(z,z)D(z,z) + (r(z)-\rho)C(z) - \int_{-\infty}^{z} C(\mu,z)S(\mu,z)T(\mu,z)d\mu$$

by using $\xi = p = S(\mu,z)$ and $\phi(\mu,z) = \phi(z,z) = r + \rho$, one gets

$$\dot{C}(z) = (r-\rho)C(z) - p(r+\rho)R(z)$$

which is the Blanchard's result.

- *The law of motion of human wealth.* In our case,

$$\begin{aligned}
\dot{D}(z) &= \xi D(z,z) - \int_{-\infty}^{z} w(z)T(\mu,z)d\mu \\
&+ \int_{-\infty}^{z}\int_{z}^{\infty} w(v)[r(z)+S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds}dvT(\mu,z)d\mu \\
&- \int_{-\infty}^{z} D(\mu,z)T(\mu,z)S(z-\mu)d\mu
\end{aligned}$$

While in Blanchard we have,

$$\dot{D}(z) = [r+p]D(z) - w(z)$$

We shall show now to which extent our equation is a very broad generalization of Blanchard's.

a) First of all, we will prove that

$$\int_{-\infty}^{z} D(\mu,z)T(\mu,z)S(z-\mu)d\mu = \xi D(z,z)$$

in the Blanchard case. Assumption (i) is fundamental. Indeed, since wages are cohort-independent, human wealth are also cohort-independent as survival probabilities are age-independent.

$$\begin{aligned}
D(\mu,z) &= \int_{z}^{\infty} w(v)e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv \\
&= \int_{z}^{\infty} w(v)e^{-\int_z^v [r(s)+p]ds}dv
\end{aligned} \tag{31}$$

which is independent of μ. Therefore, $D(\mu,z) = D(z)$ in particular. As a result,

$$\begin{aligned}
\int_{-\infty}^{z} D(\mu,z)T(\mu,z)S(z-\mu)d\mu - \xi D(z,z) &= \int_{-\infty}^{z} D(z,z)T(\mu,z)p\ d\mu - pD(z,z) \\
&= pD(z,z)\left[\int_{-\infty}^{z} T(\mu,z)d\mu - 1\right] \\
&= 0
\end{aligned}$$

because $\int_{-\infty}^{z} T(\mu,z)d\mu = 1$ by assumption.

b) The remaining terms in our equation for the dynamic of D are:

$$
\begin{aligned}
\dot{D}(z) &= \int_{-\infty}^{z}\int_{z}^{\infty} w(v)[r(z)+S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds} dv T(\mu,z)d\mu \\
&- \int_{-\infty}^{z} w(z)T(\mu,z)d\mu.
\end{aligned}
$$

Clearly,

$$
\int_{-\infty}^{z} w(z)T(\mu,z)d\mu = w(z)\int_{-\infty}^{z} T(\mu,z)d\mu = w(z)
$$

by assumptions and (ii). Moreover, the first term of the equation above can be easily rewritten with assumption (ii) as:

$$
(r(z)+p)\int_{-\infty}^{z} D(\mu,z)T(\mu,z)dz = (r(z)+p)D(z)
$$

which implies that under assumptions (i) and (ii), our D-equation (31) degenerates into Blanchard's.

- *The law of motion of nonhuman wealth.*

$$
\dot{R}(z) = R(z,z)T(z,z) + r(z)R(z) + \int_{-\infty}^{z} w(z)T(\mu,z)d\mu - C(z)
$$

With $R(z,z)=0$ and $\int_{-\infty}^{z} w(z)T(\mu,z)d\mu = w(z)\int_{-\infty}^{z} T(\mu,z)d\mu = w(z)$, we obtain

$$
\dot{R}(z) = rR(z) + w(z) - C(z)
$$

In fine, by relaxing assumptions (i) and (ii), we are able to take into account new and important demographic and economic facts, like human wealth generation-specificity or age-dependent death probability. The first key aspect is not studied in Faruquee(2003), and the second one is only explored in the life cycle perspective by the same author. We do think that our analytical exploration is necessary to take the Blanchard-Yaari model to more realistic economic demography setting (in a very broad sense).

6 The balanced growth paths

Integrating the Euler equation (24) (with $R(\mu,\mu)=0$), then replacing the obtained formula for $c(\mu,z)$ in the definition of aggregate consumption, one gets once Proposition 5 is used for an explicit representation of $c(\mu,\mu)$:

$$
\begin{aligned}
C(z) &= \int_{-\infty}^{z} \phi(\mu,\mu)D(\mu,\mu)e^{\int_\mu^z [r-\rho]ds} T(\mu,z)d\mu \\
&= \int_{z-A_{\max}}^{z} \phi(\mu,\mu)D(\mu,\mu)e^{\int_\mu^z [r-\rho]ds} T(\mu,z)d\mu
\end{aligned}
$$

where

$$D(\mu, \mu) = \int_{\mu}^{\mu + A_{\max}} \omega(v) e^{-\int_{\mu}^{v} [r(s) + S(s-\mu)] ds} dv$$

and

$$w(z) = (1 - \varepsilon) B K L^{-\varepsilon} = \bar{G} K$$

with $\bar{G} = (1 - \varepsilon) B L^{-\varepsilon}$. In order to have an explicit characterization of $C(z)$, we therefore need explicit forms for $L(z)$ and $D(\mu, \mu)$. This is done hereafter. First note that because we assume zero demographic growth in our model, the labor force is constant and equal to:

$$L = \int_{z - A_{\max}}^{z} \xi e^{n\mu} \frac{e^{\beta(\mu, z-\mu)} - \alpha(\mu)}{1 - \alpha(\mu)} d\mu = \frac{\xi}{1 - \alpha} \left[\frac{1 - e^{-\beta A_{\max}}}{\beta} - \alpha A_{\max} \right]$$

As to $D(\mu, \mu)$, we can easily refine its expression along the steady state. Since we are looking for exponential solutions for K at rate g, say $K(z) = \bar{K}\, e^{gz}$, with $\bar{K}$ a constant, we obtain:

$$\begin{aligned}
D(\mu, \mu) &= \bar{G}\bar{K} \int_{\mu}^{\mu + A_{\max}} e^{gv} e^{-\int_{\mu}^{v} [r + S(s-\mu)] ds} dv \\
&= \bar{G}\bar{K} \int_{\mu}^{\mu + A_{\max}} e^{gv} e^{-r(v-\mu) + \ln \frac{e^{-\beta(v-\mu)} - \alpha}{e^{-\beta(\mu-\mu)} - \alpha}} dv \\
&= \bar{G}\bar{K} \int_{\mu}^{\mu + A_{\max}} e^{gv} e^{-r(v-\mu)} \frac{e^{-\beta(v-\mu)} - \alpha}{e^{-\beta(\mu-\mu)} - \alpha} dv \\
&= \frac{\bar{G}\bar{K} e^{g\mu}}{1 - \alpha} \left[\frac{e^{A_{\max}(g - r - \beta)} - 1}{g - r - \beta} - \frac{\alpha (e^{A_{\max}(g-r)} - 1)}{g - r} \right] \\
&= \bar{K}\bar{P} e^{g\mu}
\end{aligned}$$

with $\bar{P} = \frac{\bar{G}}{1-\alpha} \left[\frac{e^{A_{\max}(g-r-\beta)} - 1}{g - r - \beta} - \frac{\alpha(e^{A_{\max}(g-r)} - 1)}{g - r} \right]$. Then aggregate consumption can be much more finely characterized as:

$$\begin{aligned}
C(z) &= \xi \phi(\mu, \mu) \bar{K}\bar{P} \int_{z - A_{\max}}^{z} e^{g\mu} e^{(r-\rho)(z-\mu)} \frac{e^{-\beta(z-\mu)} - \alpha}{1 - \alpha} d\mu \\
C(z) &= \frac{\xi \phi(\mu, \mu) \bar{K}\bar{P} e^{gz}}{1 - \alpha} \left[\frac{1 - e^{A_{\max}(r - \beta - \rho - g)}}{g - r + \beta + \rho} - \frac{\alpha (1 - e^{A_{\max}(r - \rho - g)})}{g - r + \rho} \right] \qquad (32)
\end{aligned}$$

Now let use that since we are along a balanced growth path, we can parameterize $C(z)$ as follows $C(z) = \bar{C}\, e^{gz}$, with $\bar{C}$ a constant. Denote $\frac{\bar{C}}{\bar{K}} = X$, then Equation (32) allows to identify X as a function of g:

$$X = \frac{\xi \phi(\mu, \mu) \bar{P}}{1 - \alpha} \left[\frac{1 - e^{A_{\max}(r - \beta - \rho - g)}}{g - r + \beta + \rho} - \frac{\alpha (1 - e^{A_{\max}(r - \rho - g)})}{g - r + \rho} \right] \qquad (33)$$

We only need another equation in terms of X and g to identify both, and this equation is simply the resource constraint of the economy. Then, combining (33) and (16), one can single out a scalar equation involving only the growth rate g:

$$F(g, \Phi) = BL^{1-\varepsilon} - g - \frac{\xi \phi(\mu, \mu) \bar{P}}{1 - \alpha} \left[\frac{1 - e^{A_{\max}(r - \beta - \rho - g)}}{g - r + \beta + \rho} - \frac{\alpha (1 - e^{A_{\max}(r - \rho - g)})}{g - r + \rho} \right] = 0 \qquad (34)$$

which allows to state the following fundamental proposition:

Proposition 10 *If $g > 0$ exists, then g solves the equation:*

$$F(g; \Phi) = 0 \tag{35}$$

where Φ is the set of parameters, $\Phi = (\alpha; \beta; \xi; \rho; \varepsilon; B)$

Unfortunately, the g-equation obtained from relation (35) is extremely complicated, specially when compared with the counterpart equation in similar models with the Blanchard demographic structure. It has been impossible for us to establish necessary and sufficient existence conditions, and uniqueness is out of analytical scope. The next proposition exhibits a sufficient condition for the g-equation to admit at least a strictly positive root.

Proposition 11 *If L or B large enough, then $g > 0$ solution to $F(.) = 0$ exists*

Proof. See appendix. ■

The sufficient condition is rather standard in economic theory: as in the original Romer (1986) model , a large enough labor force L and/or a large enough productivity parameter B are sufficient to obtain positively sloped balanced growth paths. So in a sense, and since our model relies partly on Romer's specifications, it is good news. Unfortunately, it has been impossible to bring out any analytical appraisal of the uniqueness issue. As one can see, our nonlinear equation $F(g, .) = 0$ is terrific: have in mind that even the single term $\bar{P}$ is a complicated function of g ! So studying uniqueness analytically is simply unbearable. Instead, we resort to numerical simulations with hundreds of sensitivity tests. In all the considered (numerous) parameterizations, the g-equation has a unique strictly positive solution. Then, we studied how this solution varies when the demographic parameters change both theoretically and numerically.

7 The relationship between economic growth and longevity explored

We start with an analytical result showing that in contrast to the benchmark growth model with perpetual youth, the relationship between growth and longevity cannot be strictly concave. We prove that it should be convex for low values of life expectancy and surely concave if life expectancy is large enough.

Proposition 12 *For α small enough, g is an increasing convex function of α. In contrast, g is necessarily a concave function of α when this parameter is large enough.*

Proof. The second part of the proposition is intuitive. It simply derives from the fact that since the long-run growth rate g is bounded, the increment of g following an increase in α should start decreasing after some value of α large enough. Indeed by (16), we can deduce that $g \leq B$, with B a productivity parameter independent of α. This implies that the growth rate of g should turn to negative (or zero) when α keeps growing, which disqualifies any strict convexity for large α values.

The first part of the proposition is definitely much trickier and its detailed proof is reported in the appendix. ■

Figures 11 to 14 complete and illustrate our proposition. There are principally two findings.

Insert Figures 11 to 14

1. In all our experiments, the growth rate g is an increasing function of longevity. When either α or β increases, the economic growth rate also increase. Recall the mechanisms at work. In our model, an individual saves definitely much less when old compared to her youth. Therefore, if economic development relies on accumulation of physical capital as in our model, and if such an accumulation is only possible thanks to domestic savings, then a larger longevity has also a negative effect on growth by increasing the proportion of people with relatively small saving rates. Our simulations show that at least for the set of reasonable parameterizations considered such a negative impact of increasing longevity is not enough to offset its positive contributions to growth.

2. Nonetheless, one can notice that the shape of the growth rate g as a function is mostly convex-concave, which is consistent with our empirical study, and specially with Figures 1 and 2. One may notice that such a property does not appear in Figure 13, the shape is globally concave. However, it should be noted that in this figure, the maximal age ranges from 109 to 217, and life expectancy ranges from 64 to 145, and convexity only appears when these longevity measures are low enough. Such a claim is reinforced by our Figure 14 which has a shape very similar to the estimated relationship in Figures 1 and 2. What it is the rationale behind? Well, the story is quite simple: when life expectancy is low (and economic growth is low), a further increase in life expectancy is likely to be effective in raising growth through an increment in aggregate savings, which explains why the curve is convex for low values of α. However, if life expectancy is already high, the increment in growth resulting from a further increase in life expectancy is likely to be softened because of the important proportion of elderly whose saving rates are low.

8 Conclusion

In this paper, we have studied the relationship between economic growth and longevity in a model with different demographic structures and with endogenous growth. We have started with a nonparametric econometric appraisal of this relationship on historical data showing a globally increasing but convex-concave shape. We show that life-cycle behavior combined with age-dependent survival laws can reproduce such an empirical finding. In our theory, while the economic growth rate is an increasing function of the life expectancy parameter α, its first-order derivative is non-monotonic, reflecting the growth enhancing effect of longevity at low levels of development and longevity.

An interesting extension of our framework is to endogenize life expectancy via public and private health expenditures. This could be done by considering that either parameter α or β (or both) does depend on such expenditures. This would provide a richer (and more realistic picture) of the relationship between life expectancy and economic growth. Admittedly, a substantial part of the rise in longevity registered in the twentieth century is due to rising (and more efficient) health expenditures, which was in turn made possible by better economic conditions. Incorporating health expenditures in our set-up will then result in a better and more precise appraisal of the relationship between longevity and economic growth. Unfortunately, such an extension is far from trivial as it involves (notably via the endogenization of β) further mathematical difficulties (like endogenous discounting, time inconsistency,...), which are not that easy to tackle within a vintage structure like ours.

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9 Appendix

Proof of Proposition 5

By integrating Equation (6) and the transversality condition, the budgetary constraint can be rewritten as follows

$$\begin{aligned}
\int_z^{\infty} C(\mu,v)e^{-\int_z^v [r(\in)+S(\in -z)]d\in}dv &= R(\mu,z)+D(\mu,z)\\
\int_\mu^{\infty} C(\mu,\mu)e^{\int_\mu^v (r(s)-\rho)ds} e^{-\int_\mu^v (r(\in)+S(\in-\mu))d\in}dv &= R(\mu,\mu)+D(\mu,\mu)\\
C(\mu,\mu)\int_\mu^{\infty} e^{-\int_\mu^v (\rho+S(x-\mu)dx}dv &= R(\mu,\mu)+D(\mu,\mu)\\
\Rightarrow C(\mu,\mu) &= \phi(\mu,\mu)[R(\mu,\mu)+D(\mu,\mu)]
\end{aligned}$$

with

$$\phi(\mu,\mu) = \frac{1}{\int_\mu^\infty e^{-\int_\mu^v [\rho+S(\alpha,\beta,(x-\mu))]dx}dv} \tag{36}$$

using the budgetary constraint we obtain that the quantity

$$\int_z^{\infty} C(\mu,\mu)e^{\int_\mu^z (r(s)-\rho)ds}e^{\int_z^v (r(s)-\rho)ds}e^{-\int_z^v (r(\in)+S(\in-z))d\in}dv$$

turns out to be:

$$\begin{aligned}
C(\mu,\mu)e^{\int_\mu^z (r(s)-\rho)ds}\int_z^{\infty} e^{-\int_z^v (\rho+S(s-z))ds}dv &= R(\mu,z)+D(\mu,z)\\
\Rightarrow C(\mu,z) &= \phi(\mu,z)[R(\mu,z)+D(\mu,z)] \quad \blacksquare
\end{aligned}$$

Proof of Proposition 6

We obtain:

$$\begin{aligned}
\phi(\mu,z) &= \frac{1}{\int_z^\infty e^{-\int_z^v [\rho+S(\alpha,\beta,(x-\mu))]dx}dv}\\
\phi(\mu,z)^{-1} &= \int_z^\infty e^{-\int_z^v [\rho+S(\alpha,\beta,(x-\mu))]dx}dv
\end{aligned}$$

Now, since $\int_z^v [\rho+S(\alpha,\beta,(x-\mu))]\,dx = \rho(v-z) - \ln\left(\frac{e^{-\beta(v-\mu)}-\alpha}{e^{-\beta(z-\mu)}-\alpha}\right)$, then

$$\begin{aligned}
\phi(\mu,z)^{-1} &= \int_z^\infty e^{-\rho(v-z)+\ln\frac{e^{-\beta(v-\mu)}-\alpha}{e^{-\beta(z-\mu)}-\alpha}}dv\\
&= \int_z^\infty e^{-\rho(v-z)}\frac{e^{-\beta(v-\mu)}-\alpha}{e^{-\beta(z-\mu)}-\alpha}dv\\
&= \frac{1}{e^{-\beta(z-\mu)}-\alpha}\left[\frac{e^{-\beta(z-\mu)}}{\rho+\beta}-\frac{\alpha}{\rho}\right]\\
&= \frac{\rho e^{-\beta(z-\mu)}-\alpha(\rho+\beta)}{\rho(e^{-\beta(z-\mu)}-\alpha)(\rho+\beta)}
\end{aligned}$$

Finally the marginal propensity to consume at z for an individual born at μ is:

$$\phi(\mu, z) = \frac{\rho(\rho+\beta)(e^{-\beta(z-\mu)} - \alpha)}{\rho e^{-\beta(z-\mu)} - \alpha(\rho+\beta)} \tag{37}$$

The marginal propensity to consume for a newly born individual is:

$$\phi(\mu, \mu) = \frac{1}{\int_\mu^\infty e^{-\int_\mu^v [\rho + S(\alpha,\beta,(x-\mu))]dx} dv}$$

and,

$$\begin{aligned}
\phi(\mu,\mu)^{-1} &= \int_\mu^\infty e^{-\rho(v-z)+\ln\frac{e^{-\beta(v-\mu)}-\alpha}{e^{-\beta(\mu-\mu)}-\alpha}} dv \\
&= \int_\mu^\infty e^{-\rho(v-z)} \frac{e^{-\beta(v-\mu)} - \alpha}{1-\alpha} dv \\
& \frac{-\left[e^{-(v-\mu)(\beta+\rho)}\right]_\mu^\infty}{(1-\alpha)(\rho+\beta)} + \frac{\alpha\left[e^{-\rho(v-\mu)}\right]_\mu^\infty}{\rho(1-\alpha)} \\
&= \frac{1}{(1-\alpha)(\rho+\beta)} - \frac{\alpha}{\rho(1-\alpha)} \\
&= \frac{\rho - \alpha(\rho+\beta)}{\rho(1-\alpha)(\rho+\beta)}
\end{aligned}$$

Then,

$$\phi(\mu,\mu) = \frac{\rho(1-\alpha)(\rho+\beta)}{\rho - \alpha(\rho+\beta)} \tag{38}$$

Now, let us study the evolution of $\phi(\mu,\mu)$ with respect to β, α and ρ.

$$\begin{aligned}
\frac{\partial\phi(\mu,\mu)}{\partial\beta} &= \frac{\rho(1-\alpha)\left[\rho - \alpha(\rho+\beta)\right] + \alpha\rho(1-\alpha)(\rho+\beta)}{\left[\rho-\alpha(\rho+\beta)\right]^2} \\
&= \frac{\rho^2(1-\alpha) - \alpha\rho(1-\alpha)(\rho+\beta) + \alpha\rho(1-\alpha)(\rho+\beta)}{\left[\rho-\alpha(\rho+\beta)\right]^2}
\end{aligned}$$

Then

$$\frac{\partial\phi(\mu,\mu)}{\partial\beta} = \frac{\rho^2(1-\alpha)}{\left[\rho-\alpha(\rho+\beta)\right]^2} \tag{39}$$

Since $\alpha > 1$, then $\frac{\partial\phi(\mu,\mu)}{\partial\beta} < 0$. With the expression of the maximum age we have supposed that $\beta < 0$, then an increase in β yields an increasing consumption.

$$\begin{aligned}
\frac{\partial\phi(\mu,\mu)}{\partial\alpha} &= \frac{-\rho(\rho+\beta)\left[\rho-\alpha(\rho+\beta)\right] + \rho(\rho+\beta)^2(1-\alpha)}{\left[\rho-\alpha(\rho+\beta)\right]^2} \\
&= \frac{-\rho^2(\rho+\beta) + \alpha\rho(\rho+\beta)^2 - \alpha\rho(\rho+\beta)^2 + \rho(\rho+\beta)^2}{\left[\rho-\alpha(\rho+\beta)\right]^2} \\
&= \frac{-\rho^2(\rho+\beta) + \rho(\rho+\beta)^2}{\left[\rho-\alpha(\rho+\beta)\right]^2} \\
&= \frac{\rho(\rho+\beta)\left(-\rho+\rho+\beta\right)}{\left[\rho-\alpha(\rho+\beta)\right]^2}
\end{aligned}$$

Then,

$$\frac{\partial \phi(\mu,\mu)}{\partial \alpha} = \frac{\beta\rho(\rho+\beta)}{[\rho-\alpha(\rho+\beta)]^2} \tag{40}$$

Since $\beta < 0$, then $\frac{\partial \phi(\mu,\mu)}{\partial \alpha} < 0$ if and only if $\rho + \beta > 0$. That is to say $\rho > |\beta|$.

The evolution of the marginal propensity to consume with respect to the intertemporal discount rate governed by

$$\begin{aligned}
\frac{\partial \phi(\mu,\mu)}{\partial \rho} &= \frac{[(1-\alpha)(\rho+\beta)+\rho(1-\alpha)]\,[\rho-\alpha\rho(\rho+\beta)]-\rho(1-\alpha)^2(\rho+\beta)}{[\rho-\alpha(\rho+\beta)]^2} \\
&= \frac{[(1-\alpha)(\rho+\beta)]\,[\rho-\alpha\rho(\rho+\beta)]+\rho^2(1-\alpha)-\rho(1-\alpha)(\rho+\beta)}{[\rho-\alpha(\rho+\beta)]^2}
\end{aligned}$$

Then,

$$\frac{\partial \phi(\mu,\mu)}{\partial \rho} = \frac{(1-\alpha)\left[\rho^2-\alpha(\rho+\beta)^2\right]}{[\rho-\alpha(\rho+\beta)]^2} \tag{41}$$

An increasing ρ means that agent prefer the present consumption; then they have a decreasing saving rate. That is to say $\frac{\partial \phi(\mu,\mu)}{\partial \rho} > 0$. For this purpose, we must have $\rho^2 - \alpha(\rho+\beta)^2 < 0$, since $\alpha > 1$. Finally, $\rho^2 - \alpha(\rho+\beta)^2 < 0 \Rightarrow \rho^2 < \alpha(\rho+\beta)^2$ and $\alpha > \frac{\rho^2}{(\rho+\beta)^2}$. ■

Proof of Proposition 8

We have,

$$\begin{aligned}
D(z) &= \int_{-\infty}^{z} D(\mu,z)T(\mu,z)d\mu \qquad \text{with} \quad D(\mu,z) = \int_z^{\infty} w(v)e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv \\
\dot{D}(z) &= \xi D(z,z) + \int_{-\infty}^{z} \dot{D}(\mu,z)T(\mu,z)d\mu + \int_{-\infty}^{z} D(\mu,z)\dot{T}(\mu,z)d\mu \\
\dot{D}(\mu,z) &= -w(z) + \int_z^{\infty} \dot{w}(v)e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv + \\
&\quad \int_z^{\infty} w(v)[r(z)+S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv
\end{aligned}$$

Using $\dot{T}(\mu,z) = -S(z-\mu)T(\mu,z)$ the above relation is computed as

$$\begin{aligned}
\dot{D}(z) &= \xi D(z,z) + \int_{-\infty}^{z} \left[-w(z) + \int_z^{\infty} \dot{w}(v)e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv\right. \\
&+ \left.\int_z^{\infty} w(v)[r(z)+S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds}dv\right] T(\mu,z)d\mu \\
&+ \int_{-\infty}^{z} D(\mu,z)\dot{T}(\mu,z)d\mu
\end{aligned}$$

which implies that,

$$\begin{aligned}
\dot{D}(z) &= \xi D(z,z) - w(z)L(z) \\
&+ \int_{-\infty}^{z}\int_{z}^{\infty} w(v)[r(z)+S(z-\mu)]e^{-\int_z^v [r(s)+S(s-\mu)]ds} dv T(\mu,z)d\mu \\
&- \int_{-\infty}^{z} D(\mu,z)T(\mu,z)S(z-\mu)d\mu \quad \blacksquare
\end{aligned}$$

Proof of Proposition 9

We have,

$$\begin{aligned}
R(z) &= \int_{-\infty}^{z} R(\mu,z)T(\mu,z)d\mu \\
\dot{R}(z) &= R(z,z)T(z,z) + \int_{-\infty}^{z} \dot{R}(\mu,z)T(\mu,z)d\mu + \int_{-\infty}^{z} R(\mu,z)\dot{T}(\mu,z)d\mu \\
&= R(z,z)T(z,z) + \int_{-\infty}^{z} \dot{R}(\mu,z)T(\mu,z)d\mu - \int_{-\infty}^{z} R(\mu,z)S(\mu,z)T(\mu,z)d\mu
\end{aligned}$$

Using the budget constraint given by the equation

$$\begin{aligned}
\dot{R}(z) &= R(z,z)T(z,z) + \int_{-\infty}^{z} \left[(r(z)+S(\mu,z))R(\mu,z) + w(z) - c(\mu,z)\right] T(\mu,z)d\mu \\
&- \int_{-\infty}^{z} R(\mu,z)S(\mu,z)T(\mu,z)d\mu
\end{aligned}$$

we obtain

$$\dot{R}(z) = R(z,z)T(z,z) + r(z)R(z) + w(z)L(z)d\mu - C(z) \quad \blacksquare$$

Proof of Proposition 11

We have $\lim_{g\to-\infty} F(g) = -\infty$ and,

$$\begin{aligned}
F(0) &= BL^{1-\alpha} - \frac{\xi\phi(\mu,\mu)(1-\varepsilon)BL^{-\varepsilon}\frac{1}{1-\alpha}\left[\frac{e^{A_{\max}(-r-\beta)}-1}{-r-\beta} - \frac{\alpha(e^{A_{\max}(-r)}-1)}{-r}\right]}{1-\alpha} \\
&\times \left[\frac{1-e^{A_{\max}(r-\beta-\rho)}}{-r+\beta+\rho} - \frac{\alpha(1-e^{A_{\max}(r-\rho)})}{-r+\rho}\right]
\end{aligned}$$

$$\begin{aligned}
\lim_{B,L\to\infty} F(0) &= \lim_{B,L\to\infty} BL^{1-\varepsilon} - \xi\phi(\mu,\mu)1-\varepsilon BL^{-\varepsilon}\left[\frac{e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\beta}}{-(1-\alpha)^2\varepsilon BL^{1-\varepsilon}} + \frac{\alpha e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}}{(1-\alpha)^2\varepsilon BL^{1-\varepsilon}}\right] \\
&\times \left[\frac{-e^{A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\beta-\rho}}{-\varepsilon BL^{1-\varepsilon}} - \frac{\alpha e^{A_{\max}\varepsilon BL^{1-\varepsilon}}e^{-\rho}}{\varepsilon BL^{1-\varepsilon}}\right] \\
&= \lim_{B,L\to\infty} BL^{1-\varepsilon} - \xi\phi(\mu,\mu)1-\varepsilon BL^{-\varepsilon}\left[\frac{-e^{-A_{\max}\varepsilon BL^{1-\varepsilon}}}{(1-\alpha)^2\varepsilon BL^{1-\varepsilon}}\left(e^{-\beta}-\alpha\right)\right] \\
&\times \left[\frac{e^{A_{\max}\varepsilon BL^{1-\varepsilon}}}{\varepsilon BL^{1-\varepsilon}}(e^{-\beta-\rho}-\alpha e^{-\rho})\right]
\end{aligned}$$

$$
\begin{aligned}
\lim_{B,L\to\infty} F(0) &= \lim_{B,L\to\infty} BL^{1-\varepsilon} + \xi\phi(\mu,\mu)1 - \varepsilon BL^{-\varepsilon}\left[\frac{e^{A_{\max}\varepsilon BL^{1-\varepsilon}(1-1)}}{(1-\alpha)^2\left(\varepsilon BL^{1-\varepsilon}\right)^2}\right]\left(e^{-\beta}-\alpha\right)\left(e^{-\beta-\rho}-\alpha e^{-\rho}\right) \\
&= \lim_{B,L\to\infty} BL^{1-\varepsilon} + \left[\frac{\xi\phi(\mu,\mu)(1-\varepsilon)BL^{-\varepsilon}\left(e^{-\beta}-\alpha\right)\left(e^{-\beta-\rho}-\alpha e^{-\rho}\right)}{(1-\alpha)^2\left(\varepsilon BL^{1-\varepsilon}\right)^2}\right] \\
&= \lim_{B,L\to\infty} BL^{1-\varepsilon} + \left[\frac{\xi\phi(\mu,\mu)(1-\varepsilon)\left(e^{-\beta}-\alpha\right)\left(e^{-\beta-\rho}-\alpha e^{-\rho}\right)}{(1-\alpha)^2\varepsilon^2 BL^{2-\varepsilon}}\right]
\end{aligned}
$$

Then,

$$
\lim_{B\to\infty} F(0) = +\infty
$$

and,

$$
\lim_{L\to\infty} F(0) = +\infty \text{ since } \varepsilon < 1 \quad \blacksquare
$$

Proof of Proposition 12

We have,

$$
\begin{aligned}
F &= BL^{1-\varepsilon} - g + \frac{\xi\phi(\mu,\mu)\bar{G}}{(1-\alpha)^2 D(g)}\left[\left(e^{A_{\max}(g-r-\beta)}-1\right)(g-r) - \alpha\left(e^{A_{\max}(g-r)}-1\right)(g-r-\beta)\right] \\
&\quad \times\left[(g-r+\rho)\left(1-e^{A_{\max}(r-\beta-\rho-g)}\right) - \alpha\left(1-e^{A_{\max}(r-\rho-g)}\right)(g-r+\beta+\rho)\right]
\end{aligned}
$$

Since $\lim_{\alpha\longrightarrow 1} A_{\max} = 0$ then we can approximate $e^{A_{\max}(.)}$ by $1 + A_{\max}(.)$, then the function F can be rewritten as

$$
F = BL^{1-\varepsilon} - g + \frac{\xi\phi(\mu,\mu)\bar{G}}{D(g)} A_{\max}^2 (g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)
$$

With $D(g) = g^4 + g^3(-4r+2\rho) + g^2\left[r(5r+\beta-4\rho-2) - \beta(\beta+2\rho-1) + 2\rho\right] + g(-2r+2\rho+\beta)(-2r+\beta+r^2+\beta r) + (r^2+\beta r)(-2r+2\rho+\beta)$

$R = (g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g)$

$S = \left(\frac{D'(g)}{D(g)}(g-r)-1\right)(g-r-\beta)(g-r+\rho)(r-\rho-\beta-g) - (g-r)(r-\rho-\beta-g)(-2g+2r-2\rho-\beta) + (g-r+\rho)(g-r-\beta)(g-r)$ Moreover, we can rewrite F as

$$
F = \left(BL^{1-\varepsilon}-g\right)D(g) + \xi\phi(\mu,\mu)\bar{G}A_{\max}^2(g-r-\beta)(g-r)(r-\rho-g)(r-\beta-\rho-g) \quad (42)
$$

Now, observe that if $\lim_{\alpha\longrightarrow 1} A_{\max} = 0$, then either L or g tend to zero. Consequently we can just consider the polynomes of degree one in g as first-order approximations. From the Equation

(42) we have

$$
\begin{aligned}
A_{\max}^{2} &= \frac{\left(BL^{1-\varepsilon}-g\right)D(g)}{\xi\phi(\mu,\mu)\bar{G}(g-r-\beta)\,(g-r)\,(r-\rho-g)(r-\beta-\rho-g)} \\
&= \frac{\left(BL^{1-\varepsilon}-g\right)(r^{2}+\beta r)(-2r+2\rho+\beta)}{\xi\phi(\mu,\mu)\bar{G}(g-r-\beta)\,(g-r)\,(r-\rho-g)(r-\beta-\rho-g)} \\
&\quad+\frac{BL^{1-\varepsilon}g(-2r+2\rho+\beta)(-2r+\beta+r^{2}+\beta r)}{\xi\phi(\mu,\mu)\bar{G}(g-r-\beta)\,(g-r)\,(r-\rho-g)(r-\beta-\rho-g)} \\
A_{\max}^{2} &\approx \xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}(r^{2}+\beta r)^{2}(-2r+2\rho+\beta)(r^{2}-r(\beta+2\rho)+\rho^{2}+\beta\rho) \\
&\quad+g\xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}\left[(-2r-\beta)\left[(r^{2}-r(\beta+2\rho)+\rho^{2}+\beta\rho)+r^{2}+\beta r\right]\times\right. \\
&\quad\left[(r^{2}+\beta r)(-2r+2\rho+\beta)\right]+2\left[2r(r-\beta-\rho)+\beta+\rho\right]\times \\
&\quad\left.\left[(r^{2}+\beta r)(-r^{2}+r(\beta+2\rho)-\rho^{2}-\beta\rho)\right]\right] \\
A_{\max}^{2} &\approx \xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}(r^{2}+\beta r)^{2}(-2r+2\rho+\beta)(r^{2}-r(\beta+2\rho)+\rho^{2}+\beta\rho) \\
&\quad+g\xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}M
\end{aligned}
$$

where $M=(-2r-\beta)\left[\left(r^{2}-r(\beta+2\rho)+\rho^{2}+\beta\rho\right)+r^{2}+\beta r\right]\times$
$\left[\left(r^{2}+\beta r\right)(-2r+2\rho+\beta)\right]+2\left[2r\,(r-\beta-\rho)+\beta+\rho\right]\left[(r^{2}+\beta r)(r^{2}-r(\beta+2\rho)+\rho^{2}+\beta\rho)\right]$

We can determine now the function $g(A_{\max})$:

$$
\begin{aligned}
g(A_{\max}) &= \frac{A_{\max}^{2}}{\xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}M} \\
&\quad-\frac{(r^{2}+\beta r)^{2}(-2r+2\rho+\beta)(-r^{2}+r(\beta+2\rho)-\rho^{2}-\beta\rho)}{M}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
g(A_{\max}) &= \frac{A_{\max}^{2}}{\xi\phi(\mu,\mu)\bar{G}BL^{1-\varepsilon}M} \\
&= \frac{A_{\max}^{2}}{\xi\phi(\mu,\mu)(1-\varepsilon)L^{-\varepsilon}B^{2}L^{1-\varepsilon}M} \\
&= \frac{A_{\max}^{2}}{\xi\phi(\mu,\mu)(1-\varepsilon)B^{2}\left[\frac{\xi}{1-\alpha}\left[\frac{1-e^{-\beta A_{\max}}}{\beta}-\alpha A_{\max}\right]\right]^{1-2\varepsilon}M} \\
&= \frac{\left(A_{\max}\right)^{2\varepsilon+1}}{\xi^{(2-2\varepsilon)}\phi(\mu,\mu)(1-\varepsilon)B^{2}M}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial g(A_{\max})}{\partial A_{\max}} &= \frac{(2\varepsilon+1)\left(A_{\max}\right)^{2\varepsilon}}{\xi^{(2-2\varepsilon)}\phi(\mu,\mu)(1-\varepsilon)B^{2}M} \\
\frac{\partial^{2} g(A_{\max})}{\partial A_{\max}^{2}} &= \frac{2\varepsilon\,(2\varepsilon+1)\left(A_{\max}\right)^{2\varepsilon-1}}{\xi^{(2-2\varepsilon)}\phi(\mu,\mu)(1-\varepsilon)B^{2}M}
\end{aligned}
$$

The sign of $\frac{\partial^{2} g(A_{\max})}{\partial A_{\max}^{2}}$ depends on that of M. Since we know that since $\lim\limits_{\alpha\longrightarrow 1} A_{\max}=0$, either L, r or g goes to zero, we ultimately get $M=(2\beta+\rho)\left(\rho^{2}+\beta\rho\right)>0$. Also we can conclude that $\frac{\partial^{2} g(A_{\max})}{\partial A_{\max}^{2}}\geq 0$. ■

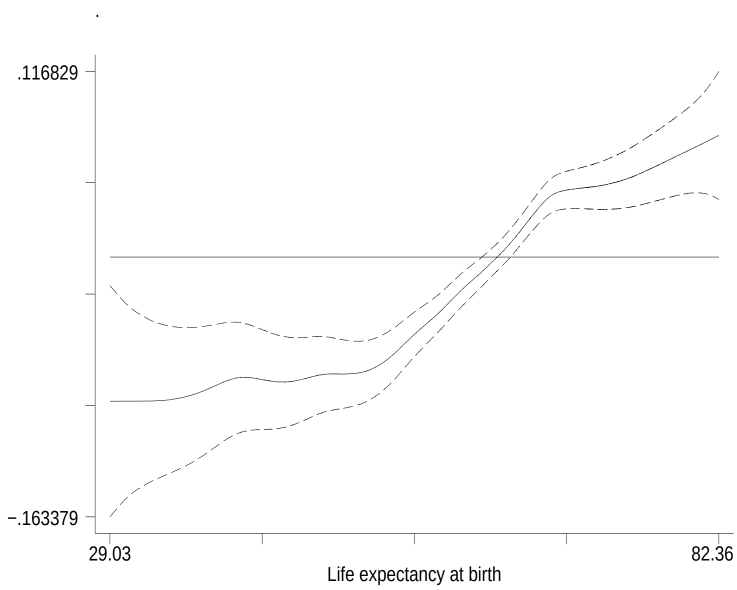

Figure 1: Nonparametric estimation of life expectancy effect on GDP growth rate per capita with yearly data. The solid line represents the nonparametric fit $\hat{\hat{f}}$. Dashed lines are 95% bootstrap pointwise confidence intervals. The straight solid line is the zero line.

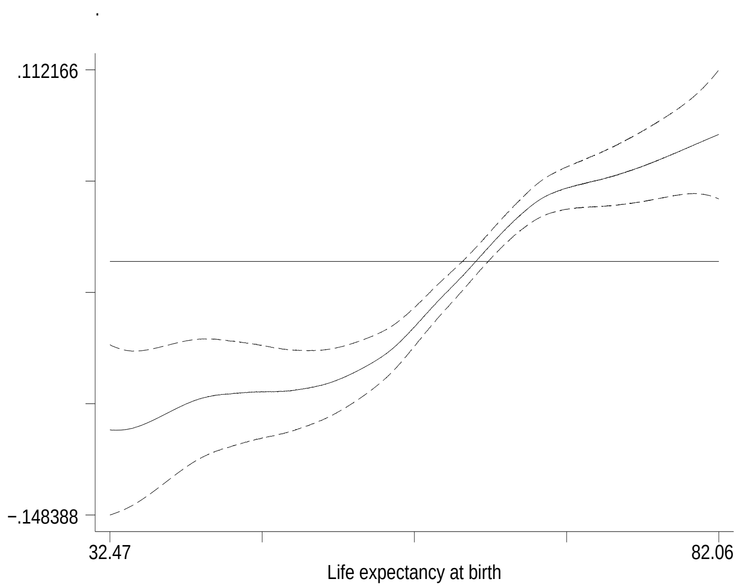

Figure 2: Nonparametric estimation of life expectancy effects on GDP growth rate per capita with 20-years average periods. The solid line represents the nonparametric fit $\hat{\hat{f}}$. Dashed lines are 95% bootstrap pointwise confidence intervals. The straight solid line is the zero line.

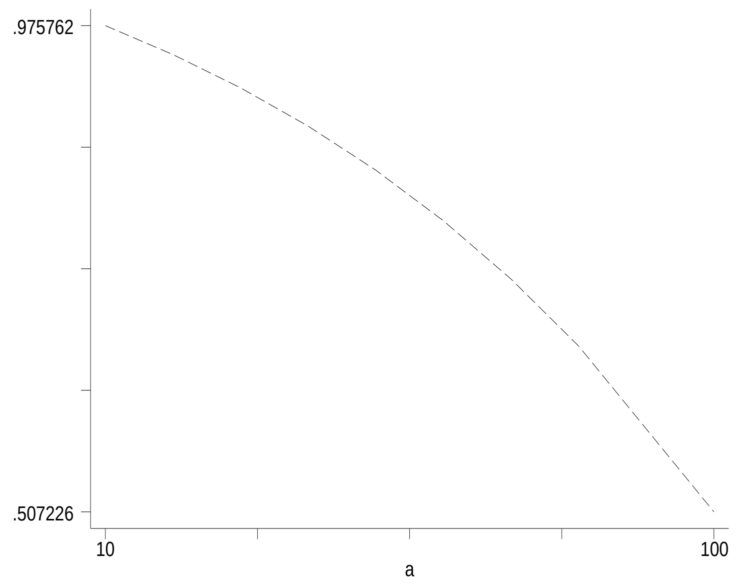

Figure 3: Evolution of survival law $m(a)$ with respect to age (a).
$\alpha = 5.44$, $\beta = -0.0147$, $E = 73$, $A_{\max} = 115$

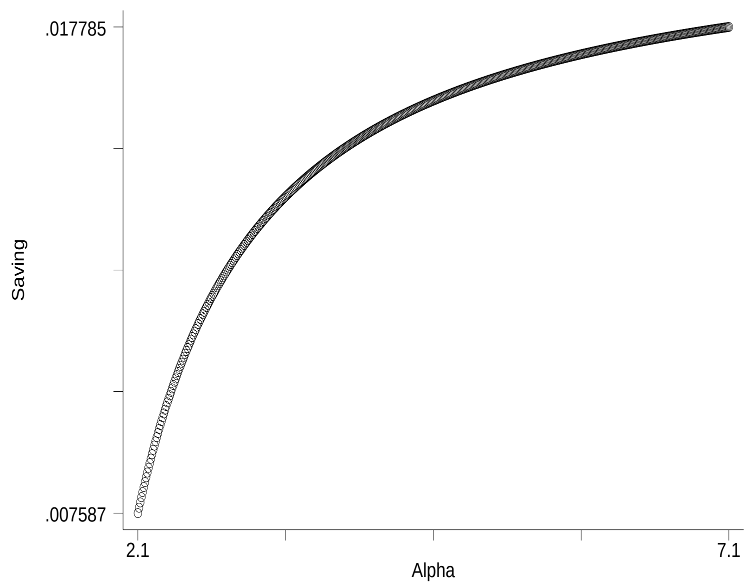

Figure 4: Evolution of $s(\mu, \mu)$ with respect to α.
$\beta = -0.01502, \rho = 0.98$ and $27.7 < E < 85$, $49 < A_{\max} < 130$

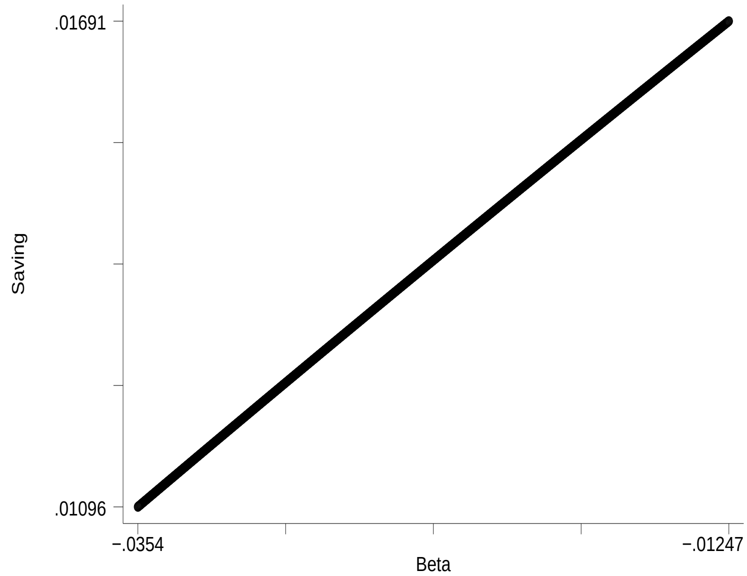

Figure 5: Evolution of $s(\mu, \mu)$ with respect to β.
$\alpha = 5.1, \rho = 0.98$ and $29.7 < E < 82$, $46 < A_{\max} < 130$

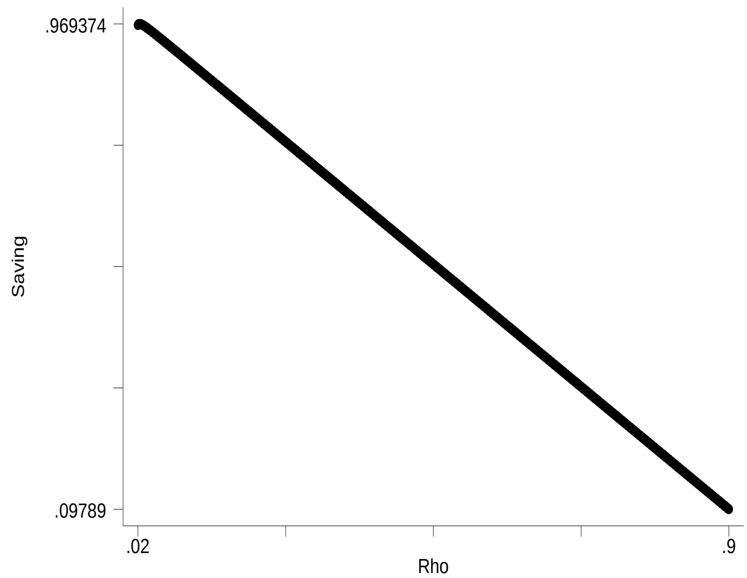

Figure 6: Evolution of $s(\mu, \mu)$ with respect to ρ.
$\alpha = 8.1, \beta = -0.0147$ and $E = 94$, $A_{\max} = 142$

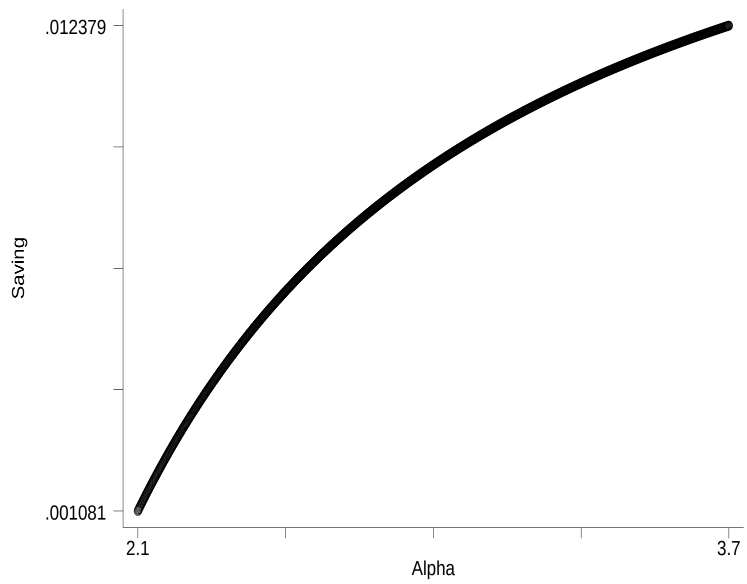

Figure 7: Evolution of $s(\mu, \mu)$ with respect to α.
$\beta = -0.02, \rho = 0.98$ and $20.8 < E < 39.6$, $37.09 < A_{\max} < 66.5$

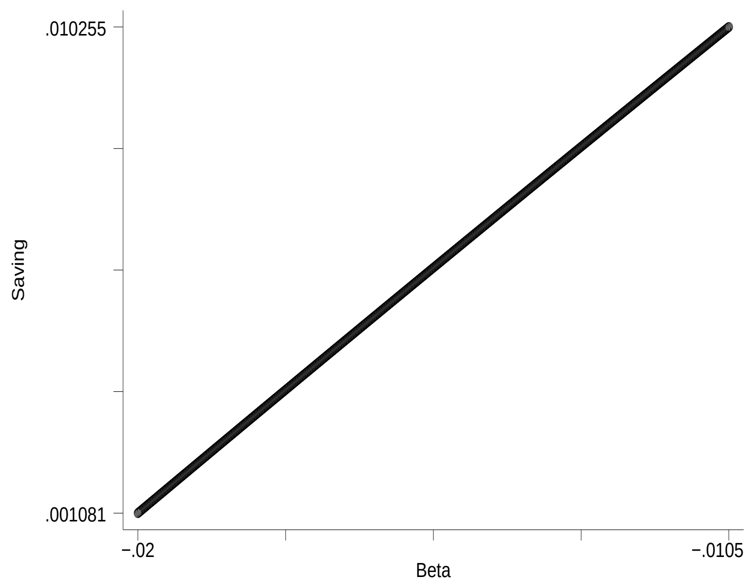

Figure 8: Evolution of $s(\mu, \mu)$ with respect to β.
$\alpha = 2.1, \rho = 0.98$ and $20.8 < E < 39.65$, $37 < A_{\max} < 70.6$

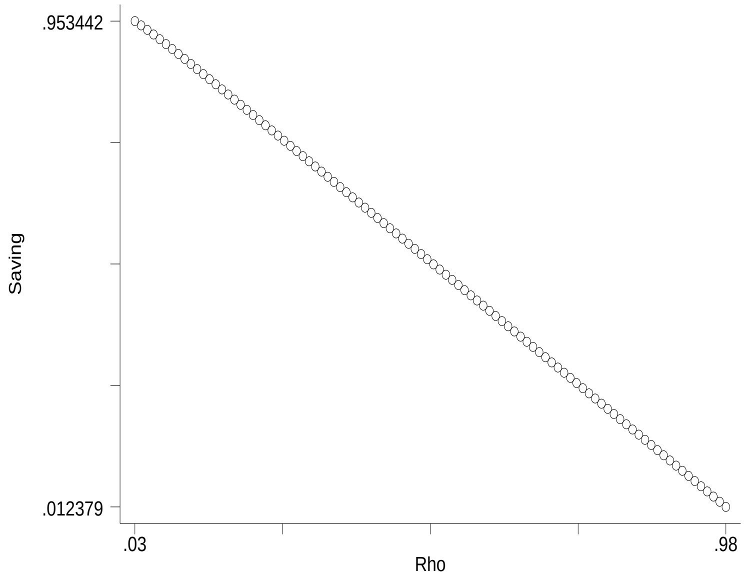

Figure 9: Evolution of $s(\mu, \mu)$ with respect to ρ.
$\alpha = 3.7, \beta = -0.02$ and $E = 39.6$, $A_{\max} = 65$

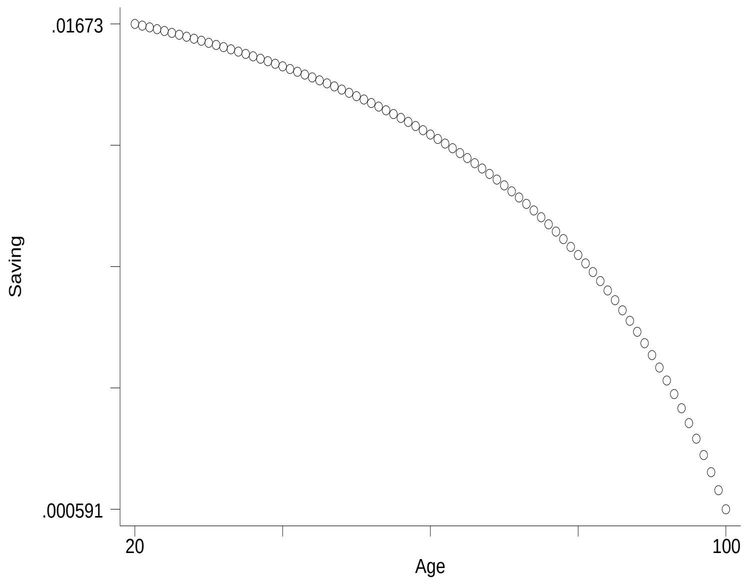

Figure 10: Evolution of saving rate with respect to a.
$\alpha = 7.1, \beta = -0.01502, \rho = 0.98$ and $E = 91$, $A_{\max} = 139$

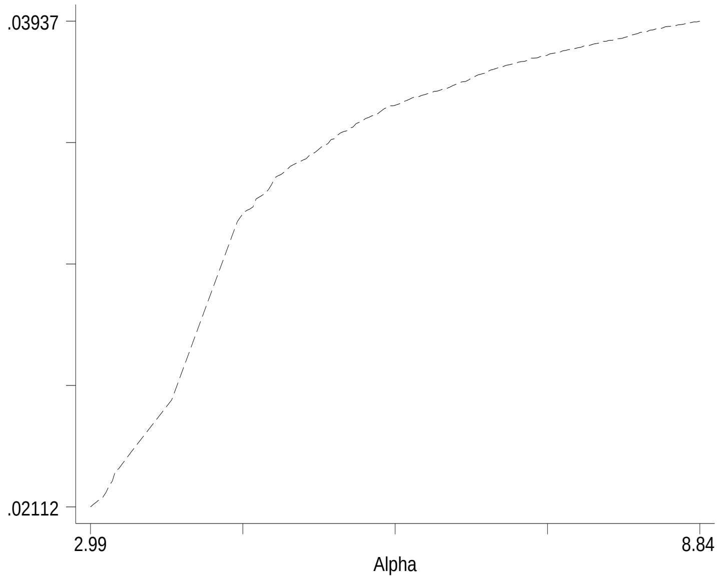

Figure 11: Evolution of g with respect to α.

$\beta = -0.015, \rho = 0.02$ and $43.03 < E < 97.23$, $73 < A_{\max} < 145.3$

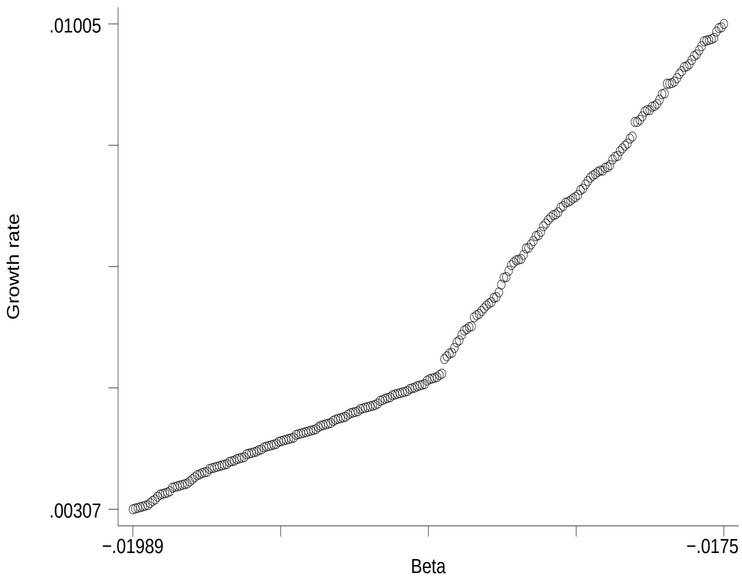

Figure 12: Evolution of g with respect to β.

$\alpha = 5.1, \rho = 0.02$ and $51 < E < 59$, $81.86 < A_{\max} < 93$

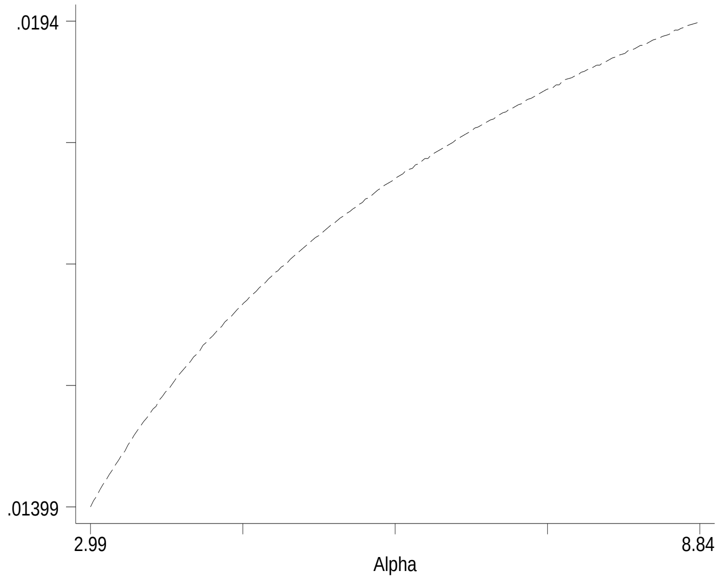

Figure 13: Evolution of g with respect to α.
$\beta = -0.01, \rho = 0.02$ and $64.6 < E < 145$, $109 < A_{\max} < 217$

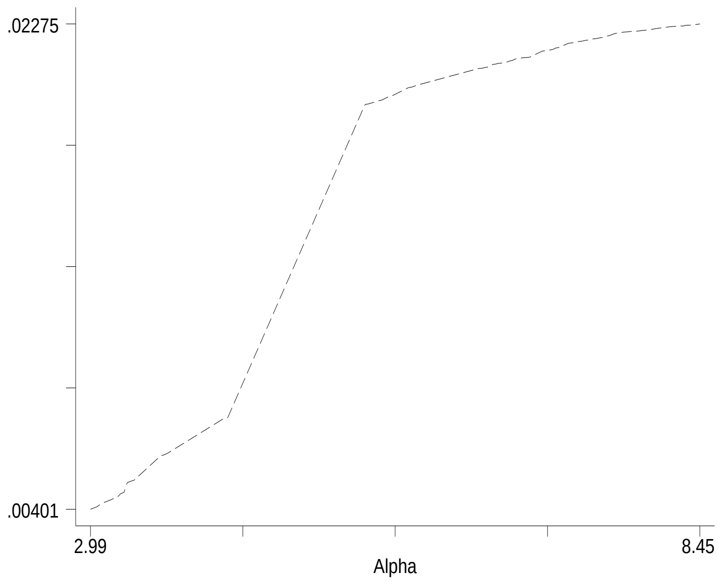

Figure 14: Evolution of g with respect to α.
$\beta = -0.017, \rho = 0.02$ and $38 < E < 85$, $64 < A_{\max} < 125$