Non-Cooperative Asymptotic Oligopoly in Economies with Infinitely Many Commodities*

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Abstract

In this paper, we extend the non-cooperative analysis of oligopoly to exchange economies with infinitely many commodities by using strategic market games. This setting can be interpreted as a model of oligopoly with differentiated commodities by using the Hotelling line. We prove the existence of an "active" Cournot-Nash equilibrium and show that, when traders are replicated, the price vector and the allocation converge to the Walras equilibrium. We examine how the notion of oligopoly extends to our setting with a coutable infinity of commodities by distinguishing between asymptotic oligopolists and asymptotic price-takers. We illustrate these notions via a number of examples.

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1 Introduction

Non-cooperative oligopoly, in a general equilibrium setting, is usually studied in economies with a finite number of commodities. In this paper, we extend the analysis of non-cooperative oligopoly to exchange economies with a countable infinity of commodities by using the strategic market game analysed by Dubey and Shubik (1978) (DS hereafter). As we make explicit in Section 3, by using the Hotelling line, our setting can be interpreted as a model of oligopoly with differentiated commodities. The strategic market game analysed by DS is a generalization of the contributions of Shubik (1973), Shapley (1976), and Shapley and Shubik (1977). In this class of games, when traders and commodities are finite, all traders turn out to have market power on all commodities. Differently, the market power of a trader, who is active on an infinite set of commodities, can converge to zero along the sequence of commodities or can remain non-negligible. To describe these new phenomena, we introduce two notions. We say that a trader is an "asymptotic oligopolist" if his market power is uniformly bounded away from zero on an infinite subset of commodities. On the contrary, if trader's market power converges to zero along the sequence of commodity, we say that the trader is an "asymptotic price-taker". The former can be simply interpreted as the extension of the classical notion of oligopolist to infinite economies (see Cournot (1838)). The latter describes a trader with a mixed behaviour since she has a non-negligible market power on a finite set of commodities while being an "approximate" price-taker[1] on an infinite set of commodities. This kind of

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[1] Approximate means that trader's market power is not zero but it can be arbitrary small by considering different infinite sets of commodities.

mixed behaviour arises endogenously, in equilibrium, in the strategic market game studied by us and cannot arise in a setting with a finite number of commodities.

In an earlier literature on imperfect competition, it is possible to find similar examples of mixed behaviour. Negishi (1961) extended the theory of monopolistic competition of Chamberlin (1933) and Robinson (1933) from partial to general equilibrium. To do so, he considered monopolistically competitive firms where each firm is characterized by a portfolio of commodities containing all the commodities on which the firm has market power. He assumed that portfolios are strict subsets of all commodities so that a monopolistically competitive firm has market power on a subset of commodities while acting competitively on other commodities. In this way, he ruled out from his analysis both monopoly and oligopoly. In a similar vein, Gabszewicz and Michel (1997) introduced the notion of portfolio in the Cournot-Walras literature on exchange economies initiated by Codognato and Gabszewicz (1991). In their model, some traders are defined oligopolists and each of them is characterized by a portfolio that is a subset of the commodities owned by the oligopolist. In all these contributions, the portfolio of commodities is a primitive of the model and no formal explanation is given as to why a particular trader should behave strategically on some commodities and competitively on others. In contrast, in our paper, by using strategic market games, a traders' market power is endogenously determined in equilibrium.

Our contributions are as follows. We first define an exchange economy with a countable infinity of commodities and traders having a structure of multilateral oligopoly, i.e., an economy in which each trader owns commodity money and only one other commodity. Our approach relies on the literature on economies with infinitely many commodities and with a double infinity of commodities and traders initiated by Bewley (1972) and Balasko, Cass, and Shell (1980) respectively. We then introduce the strategic market game and, as the previous contributions in this literature (see DS, Amir, Sahi, Shubik, and Yao (1990), and Sahi and Yao (1989), among others), we prove the existence of a Cournot-Nash equilibrium and the convergence to the Walras equilibrium. The game analysed by DS is a strategic market game in which there is a trading post for each commodity where the commodity is exchanged for commodity money. The actions available to traders are bids, amounts of commodity money given in exchange for other commodities, and offers, amounts of commodities, owned by traders, put up in exchange for commodity money. Since in this game a Cournot-Nash equilibrium with no trade always exists, we prove the existence of an "active" Cournot-Nash equilibrium,[2] at which all com- modities are exchanged. The proof adapts the approach used by Bloch and Ferrer (2001) for the case of two commodities to a setting with an infinite set of commodities. However, in a setting characterized by an infinite commodity space, additional restrictions on the marginal utilities of commodities in traders' initial endowments are also required. After having defined the model and stated the main results, we first introduce the formal definitions of "asymp- totic oligopolist" and "asymptotic price-taker". We then show some examples to illustrate the notion of asymptotic oligopolist and to provide, heuristically, under which conditions an asymptotic oligopolist exists. Perhaps surprisingly, we construct an example where, even if the number of traders active in each trading post is not uniformly bounded, there are traders "big enough", in terms of initial endowment of commodity money, who are asymptotic oligopolists.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we state the existence and the convergence theorems. In Section 3, we introduce the definitions of asymptotic oligopolist and asymptotic price-taker and we show some examples. In Section 4, we prove the two theorems. In Section 5, we draw some conclusions from our analysis. In the appendixes, we list all the mathematical definitions and results, we establish the relationship

[2]DS proved the existence of an "equilibrium point" that is a Cournot-Nash equilibrium in which some commodities are legitimately not exchanged. See Cordella and Gabszewicz (1998) and Busetto and Codognato (2006) for a detailed analysis on "legitimately inactive" trading posts.

between our model and the strategic market game analysed by DS, and we relate our market power measure with the notions of marginal price and average price introduced by Okuno, Postlewaite, and Roberts (1980).

2 The mathematical model

Let T_t be a finite set with cardinality k strictly greater than 1. Elements of T_t are traders of type t. The *set of traders* is $I = \cup_{t=1}^{\infty} T_t$. The *set of commodities* is $J = \{0, 1, 2, \dots\}$. The *commodity space* is the space of bounded sequences ℓ_∞.[3] The *consumption set* is a subset of the commodity space and it is denoted by X. A *commodity bundle* x is a point in the consumption set, where x_j denotes the amount of commodity j. Let N be a subset of natural numbers, $X_{N>0} = \{x \in X : x_j \geq 0$, for each $j \in J$ and $x_j > 0$, for each $j \in N\}$. A trader i is characterized by an initial endowment, $w^i \in X$, and a utility function, $u^i : X \to \mathbb{R}$. Traders of the same type have the same initial endowment and utility function. The context should clarify whether the superscript refers to a trader type or to a trader. An *exchange economy* is then a set $\mathcal{E} = \{(u^i, w^i) : i \in I\}$.

An *allocation* $\mathbf{x}$ is a specification of a commodity bundle x^i, for each $i \in I$, such that $\sum_{i \in I} x_j^i = \sum_{i \in I} w_j^i$, for each $j \in J$. A *price vector* is denoted by p. Given a price vector p, we define the *budget set* of trader i to be $B^i(p) = \{x \in X : \sum_{j=0}^{\infty} p_j x_j^i \leq \sum_{j=0}^{\infty} p_j w_j^i\}$. A *Walras equilibrium* is a pair $(p, \mathbf{x})$ consisting of a price vector p and an allocation $\mathbf{x}$ such that x^i is maximal with respect to u^i in i's budget set, $B^i(p)$, for each $i \in I$.

A *commodity j is desired by a trader i* if u^i is an increasing function of the variable x_j^i and $\lim_{x_j^i \to 0} \frac{\partial u^i}{\partial x_j^i} = \infty$, for any fixed choice of the other variables. The set of commodities desired by a trader i is denoted by L^i and the set of traders that desire commodity j is denoted by H_j.

We make the following assumptions.

Assumption 1. Let σ be a positive constant. The initial endowment of a type t trader is such that $w_0^t > 0$, $w_t^t > \sigma$, and $w_j^t = 0$, for each $j \in J \setminus \{0, t\}$, for $t = 1, 2, \dots$.

Assumption 2. Let e be a positive constant such that $\sigma < e$. The aggregate initial endowment of each commodity is such that $\sum_{t=1}^{\infty} w_0^t < e$ and $\sum_{t=1}^{\infty} w_j^t < e$, for each $j \in J \setminus \{0\}$.

Assumption 3. The utility function of a type t trader is continuous in the product topology,[4] continuously Frèchet differentiable in $X_{L^t>0}$, non decreasing, and concave, for $t = 1, 2, \dots$. Moreover, let λ and f be two positive constants such that $\lambda < f$, the marginal utilities of a type t trader are such that $\lambda \leq \frac{\partial u^t}{\partial x_t^t}(x^t) \leq \frac{\partial u^t}{\partial x_0^t}(x^t) \leq f$,[5] for each $x^t \in X$, for $t = 1, 2, \dots$.

Assumption 4. A commodity j is desired by at least one type of trader, for each $j \in J \setminus \{0\}$.

The first two assumptions are related to the structure of initial endowments. The first imposes the structure of multilateral oligopoly[6] and the second ensures that the aggregate initial endowment is uniformly bounded. The third assumption imposes common restrictions on the preferences of the traders. We need the additional conditions on the marginal utilities of the commodities owned by a trader because we deal with an infinite number of commodities. The last assumption is standard in the literature of strategic market game but our definition

[3] All the mathematical definitions and results can be found in Appendix A.

[4] In the space of bounded sequences, ℓ_∞, continuity in the product topology is equivalent to the continuity in the Mackey topology (See Brown and Lewis (1981)).

[5] This assumption is satisfied, for instance, by utility functions linear in the commodities 0 and t.

[6] Shubik (1973) refers to multilateral oligopolies with commodity money as "world of oligopolies".

of desired commodity is more restrictive than the one of DS because we impose additional restrictions on the limits of marginal utilities.

We now introduce the strategic market game Γ associated with the exchange economy $\mathcal{E}$.[7] For each commodity $j \in J \setminus \{0\}$, there is a trading post where commodity j is exchanged for commodity money 0. Moreover, we impose that each trader can bid only on the commodities that he does not own, i.e. there is no wash sales.

The strategy set of a trader i of type t is

$$S^i = \Big\{ s^i = (q^i_t, b^i_1, \dots, b^i_{t-1}, b^i_{t+1}, \dots) : 0 \leq q^i_t \leq w^i_t,\ b^i_j \geq 0,\ \text{for } j \in J \setminus \{0, t\}, \\ \text{and } \sum_{j \neq 0, t} b^i_j \leq w^i_0 \Big\},$$

where q^i_t is the quantity of commodity t that trader i offers to sale and b^i_j is the amount of commodity money that he bids on commodity j. Without loss of generality, we make the following technical assumption on the strategy set.

Assumption 5. The set S^i is a subset of ℓ_∞ endowed with the product topology, for each $i \in I$, i.e., $S^i \subseteq \{s^i \in \ell_\infty : \sup |s^i_j| \leq e\}$.

By assuming that the space ℓ_∞ in endowed with the product topology, we impose on ℓ_∞ the norm $\|x\|_\infty = \sup_i \|a_i x_i\|$ such that $\{a_i\}$ is a sequence of real number converging to zero. (see Brown and Lewis (1981)). This assumption ensures that S^i lies in a normed space and then in a Hausdorff space.

Let $S = \prod_{i \in I} S^i$ and $S^{-z} = \prod_{i \in I \setminus \{z\}} S^i$. Let s and s^{-i} be elements of S and S^{-i} respectively. For each $s \in S$, the price vector $p(s)$ is such that

$$p_j(s) = \begin{cases} \frac{\bar{b}_j}{\bar{q}_j} & \text{if } \bar{q}_j \neq 0 \\ 0 & \text{if } \bar{q}_j = 0 \end{cases}, \tag{1}$$

for each $j \in J \setminus \{0\}$, with $\bar{q}_j = \sum_{i \in T_j} q^i_j$ and $\bar{b}_j = \sum_{i \in I \setminus T_j} b^i_j$. By Assumption 2, the sums $\bar{q}_j$ and $\bar{b}_j$ are finite. For each $s \in S$, the commodity bundle $x^i(s)$ of a trader i of type t is such that

$$\begin{aligned} x^i_0(s) &= w^i_0 - \sum_{j \neq 0, t} b^i_j + q^i_t p_t(s), \\ x^i_t(s) &= w^i_t - q^i_t, \\ x^i_j(s) &= \begin{cases} \frac{b^i_j}{p_j(s)} & \text{if } p_j(s) \neq 0 \\ 0 & \text{if } p_j(s) = 0 \end{cases}, \text{ for each } j \in J \setminus \{0, t\}. \end{aligned}$$

The payoff function of a trader i is $\pi^i(s) = u^i(x^i(s))$.

We now introduce the definitions of an active trading post, a best response correspondence, and a Cournot-Nash equilibrium.

Definition 1. A trading post for a commodity j is said to be active if $\bar{q}_j > 0$ and $\bar{b}_j > 0$, otherwise we say that the trading post is inactive.

Definition 2. The best response correspondence of a trader i is a correspondence $\phi^i : S^{-i} \to S^i$ such that

$$\phi^i(s^{-i}) \in \arg\max_{s^i \in S^i} \pi^i(s^i, s^{-i}),$$

for each $s^{-i} \in S^{-i}$.

[7] This game was introduced by Shubik (1973). In Appendix B, we prove that in multilateral oligopolies the game Γ is equivalent to the game analysed by DS in terms of attainable commodity bundles.

Definition 3. An $\hat{s} \in S$ is a Cournot-Nash equilibrium of Γ if $\hat{s}^i \in \phi^i(\hat{s}^{-i})$, for each $i \in I$.

An *active Cournot-Nash equilibrium* is a Cournot-Nash equilibrium such that all trading posts are active. A *type-symmetric Cournot-Nash equilibrium* is a Cournot-Nash equilibrium such that all traders of the same type play the same strategy. An *interior Cournot-Nash equilibrium* is a Cournot-Nash equilibrium such that $\sum_{j \neq 0,t} b_j^i < w_0^i$, for each $i \in I$.

We now state the existence theorem.

Theorem 1. Under Assumptions 1, 2, 3, 4, and 5, an active Cournot-Nash equilibrium of Γ exists.

In the next theorem, we show that if traders are replicated infinitely many times, then the price vector and the allocation, at the Cournot-Nash equilibrium, converge to the Walras equilibrium of the underlying exchange economy.[8] Before to state the theorem, we introduce the following additional notation. Let ${}_k\Gamma$ be a game Γ with k replicas of each type of traders and ${}_k\hat{s}$ be a type symmetric Cournot-Nash equilibrium of ${}_k\Gamma$. To each ${}_k\hat{s}$ can be associated a vector ${}_k\tilde{s}$ that associate a strategy to each type of trader, i.e., ${}_k\tilde{s} \in \prod_{t=1}^{\infty} S^t$ and ${}_k\tilde{s}^t = {}_k\hat{s}^t$, for $t = 1, 2, \ldots$. We denote by $p({}_k\tilde{s})$ and ${}_h\mathbf{x}({}_k\tilde{s})$ a price vector and an allocation of an exchange economy with h replicas of each type of trader such that $p({}_k\tilde{s}) = p({}_k\hat{s})$ and ${}_hx^t({}_k\tilde{s}) = x^t({}_k\hat{s})$, for $t = 1, 2, \ldots$, for all h replicas.

Theorem 2. Consider a sequence of games $\{{}_k\Gamma\}_{k=2}^{\infty}$. Suppose there exists a sequence of interior type-symmetric active Cournot-Nash equilibria, $\{{}_k\hat{s}\}_{k=2}^{\infty}$, such that the sequences $\{{}_k\tilde{s}\}_{k=2}^{\infty}$ and $\{p({}_k\tilde{s})\}_{k=2}^{\infty}$ converge to $\tilde{v}$ and to $p(\tilde{v})$, respectively. Then, the pair $(p(\tilde{v}), {}_k\mathbf{x}(\tilde{v}))$ is a Walras equilibrium of the exchange economy associated to the game ${}_k\Gamma$, for any k.

3 Asymptotic oligopolists: patterns of strategic behaviour

In this section, we first introduce a market power measure for the game Γ and we define the notions of an asymptotic oligopolist and an asymptotic price-taker. We then focus on the notion of asymptotic oligopolist and we illustrate it with examples.

In the game Γ, the market power of a trader i of type t on commodity j, with $j \neq t$, can be measured by the following ratio $b_j^i/\bar{b}_j$. The higher this ratio is, the higher is the market power of trader i on commodity j. If $b_j^i = 0$, we say that trader i is a trivial price taker on commodity j. The market power of a trader i of type t on commodity t can be measured in the analogous way by the index $q_t^i/\bar{q}_t$.

Definition 4. Consider an active Cournot-Nash equilibrium $\hat{s}$ in which there exists a trader i such that $\hat{b}_j^i > 0$ for an infinite number of commodities. We say that trader i is an asymptotic price-taker if $\lim_{j \to \infty} \hat{b}_j^i/\bar{\hat{b}}_j = 0$, otherwise we say that trader i is an asymptotic oligopolist.

The key features of an asymptotic oligopolist are that he consumes an infinite number of commodities and his market power is uniformly bounded away from zero on an infinite set of commodities.

We now show some examples to illustrate the notion of asymptotic oligopolist and to provide, heuristically, under which conditions an asymptotic oligopolist exists. To simplify computations, in all examples we consider logarithmic additive utility functions of the form $x_0^i + x_t^i + \sum_j \beta_j \ln x_j^i$.[9] All the utility functions in the examples below are linear in commodity

[8]Our setting can be considered as a particular case of the exchange economy considered by Wilson (1981) for which a Walras equilibrium exists.

[9]Logarithmic utility functions facilitate computations but are not continuous at the boundary. Therefore they violate Assumption 3. This does not affect the current analysis but should be kept in mind.

money (the numeraire good) as well as the commodity owned by the trader. Moreover, to each commodity is associated a utility coefficient, β_j, that converts the utility associated with consumption of that commodity into the numeraire good. The use of logarithmic utility functions implies that traders can consume a large or an infinite number of commodities. This means that commodities are imperfect substitutes and traders display preference for variety. Furthermore, we assume that each commodity corresponds to a point on a Hotelling line, so that the degree of substitutability between commodities is measured by their distance. In particular, we consider an Hotelling line of length one on which commodities are placed in the following way: commodity 1 is at the origin of the line, commodity 2 is in the middle of the line, commodity 3 is in the middle of the second half, and so on. Figure 1 illustrates the arrangement of commodities along the line. As the diagram makes clear, commodities are

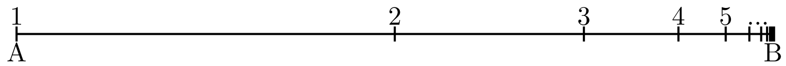

Figure 1

clustered in the vicinity of point B. At the Cournot-Nash equilibrium of each example below, all traders, who consume positive amount of an infinite number of commodities, concentrate the higher bids on the commodities at the beginning of the Hotelling line and spread the small bids on the commodities clustered around point B. The economic interpretation of this behaviour is that, heuristically, traders prefer to spread small bids on a large variety of similar commodities and then bids are higher only for the commodities with no close substitutes.[10]

In the first example, we show an exchange economy in which at the Cournot-Nash equilibrium some traders are asymptotic oligopolists.

Example 1. Consider an exchange economy in which traders of type 1, 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$u^1(x^1) = x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 \qquad w^1 = \left(\frac{1}{1-\delta}, 1, 0, \dots\right),$$

$$u^2(x^2) = x_0^2 + \delta \ln x_1^2 + x_2^2 + \delta^3 \ln x_3^2 \qquad w^2 = \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \dots\right),$$

$$u^3(x^3) = x_0^3 + x_3^3 + \delta^4 \ln x_4^3 \qquad w^3 = \left(\frac{\delta^2}{1-\delta}, 0, 0, 1, 0, \dots\right),$$

$$u^t(x^t) = x_0^t + x_t^t + \delta^{t+1} \ln x_{t+1}^t \qquad w^t = \left(\frac{\delta^{t-1}}{1-\delta}, 0, \dots, 0, 1, 0, \dots\right),$$

with $\delta \in (0,1)$. The type-symmetric active Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$\begin{aligned}
\left(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots, \hat{b}_j^1, \dots\right) &= \left(\delta^1 G(1)^2, \delta^2 G(1), \delta^3 G(2), \dots, \delta^j G(2), \dots\right),\\
\left(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \hat{b}_4^2, \dots, \hat{b}_j^2, \dots\right) &= \left(\delta^2 G(1)^2, \delta^1 G(1), \delta^3 G(2), 0, \dots, 0, \dots\right),\\
\left(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \hat{b}_5^3, \dots, \hat{b}_j^3, \dots\right) &= \left(2\delta^3 G(1)G(2), 0, 0, \delta^4 G(2), 0, \dots, 0, \dots\right),\\
\left(\hat{q}_t^t, \hat{b}_1^t, \dots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \hat{b}_{t+2}^t, \dots\right) &= \left(2\delta^t G(1)G(2), 0, \dots, 0, \delta^{t+1} G(2), 0, \dots\right),
\end{aligned}$$

for $t = 4, 5, \dots$, with $G(y) = \left(1 - \frac{1}{yk}\right)$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

[10] This interpretation also provides a possible economic meaning to continuity in the product topology in a setting with differentiated commodities.

Proof. At the Cournot-Nash equilibrium, the market power measure of type 1 traders, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{1}{2k}$, for each $j \geq 3$. Hence $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = \frac{1}{2k}$. ■

In this example, type 1 traders are the asymptotic oligopolists because all traders make the same bid on each commodity j and in each trading post "only" $2k$ traders are active.

In the next example, we show that if this latter feature is not true, no trader is an asymptotic oligopolist.

Example 2. Consider an exchange economy in which traders of type 1 are the same of Example 1 and traders of types 2, 3, and $t \geq 4$ have the following utility functions and initial endowments

$$u^2(x^2) = x_0^2 + \delta \ln x_1^2 + x_2^2 + \sum_{j=3}^{\infty} \delta^j \ln x_j^2 \qquad w^2 = \Big(\frac{\delta^1}{1-\delta}, 0, 1, 0, \dots\Big),$$
$$u^3(x^3) = x_0^3 + x_3^3 + \sum_{j=4}^{\infty} \delta^j \ln x_j^3 \qquad w^3 = \Big(\frac{\delta^2}{1-\delta}, 0, 0, 1, 0, \dots\Big),$$
$$u^t(x^t) = x_0^t + x_t^t + \sum_{j=t+1}^{\infty} \delta^j \ln x_j^t \qquad w^t = \Big(\frac{\delta^{t-1}}{1-\delta}, 0, \dots, 0, 1, 0, \dots\Big),$$

with $\delta \in (0,1)$. The type-symmetric active Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$\begin{aligned}
\big(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots, \hat{b}_j^1, \dots\big) &= \big(\delta^1 G(1)^2, \delta^2 G(1), \delta^3 G(2), \dots, \delta^j G(j-1), \dots\big),\\
\big(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \dots, \hat{b}_j^2, \dots\big) &= \big(\delta^2 G(1)^2, \delta^1 G(1), \delta^3 G(2), \dots, \delta^j G(j-1), \dots\big),\\
\big(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \dots, \hat{b}_j^3, \dots\big) &= \big(2\delta^3 G(1)G(2), 0, 0, \delta^4 G(3), \dots, \delta^j G(j-1), \dots\big),\\
\big(\hat{q}_t^t, \hat{b}_1^t, \dots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \dots\big) &= \big((t-1)\delta^t G(1)G(t-1), 0, \dots, 0, \delta^{t+1} G(t), \dots\big),
\end{aligned}$$

for $t = 4, 5, \dots$, with $G(y) = \big(1 - \frac{1}{yk}\big)$. At the Cournot-Nash equilibrium, no trader is an asymptotic oligopolist.

Proof. At the Cournot-Nash equilibrium, the market power measure of a type t trader, $\hat{b}_j^t/\bar{\hat{b}}_j$, is equal to $\frac{1}{(j-1)k}$, for each $j \geq 3$, for $t = 1, 2, \dots$. Hence, $\lim_{j\to\infty} \hat{b}_j^t/\bar{\hat{b}}_j = 0$, for $t = 1, 2, \dots$. ■

The crucial difference with Example 1 is that the number of traders active in each trading post is not uniformly bounded and this counteracts the market power of type 1 traders. Since the number of active traders in each trading post is striclty increasing and all traders make the same bid on each commodity, traders' bids become negligible in comparison to the bids of all others along the sequence of commodities. Hence, all traders are asymptotic price-takers.

A different result is obtained if there is a type of trader that places higher bids than other traders along the sequence of commodities. In the next example, we show that, even if the number of traders that desire each commodity in not uniformly bounded, there are asymptotic oligopolists.

Example 3. Consider an exchange economy in which traders of types 2, 3, and $t \geq 4$ are the same of Example 2 and traders of type 1 have the following utility function and initial endowment

$$u^1(x^1) = x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \frac{1}{j^2} \ln x_j^1, \qquad w^1 = \Big(\frac{\pi^2}{6}, 1, 0, \dots\Big).$$

For $k = 2$ and $\delta = 1/3$, the type-symmetric active Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$(\hat{q}_1^1, \hat{b}_2^1, \hat{b}_3^1, \dots, \hat{b}_j^1, \dots) = \left(\frac{1}{12}, \frac{1}{8}, \frac{\sqrt{13}-1}{36}, \dots, \frac{(2j+1)j^2 - 3^j - F(j)}{4j^2\,(j^2 - 3^j)}, \dots\right),$$

$$(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \dots, \hat{b}_j^2, \dots) = \left(\frac{1}{16}, \frac{1}{6}, \frac{7-\sqrt{13}}{108}, \dots, \frac{(5-2j)j^2 + 3^j(4j-7) - F(j)}{3^j 4(j-2)(3^j - j^2)}, \dots\right),$$

$$(\hat{q}_3^3, \hat{b}_1^3, \hat{b}_2^3, \hat{b}_4^3, \dots, \hat{b}_j^3, \dots) = \left(\frac{2+\sqrt{13}}{108}, 0, 0, \frac{227-\sqrt{4729}}{14040}, \dots, \hat{b}_j^2, \dots\right),$$

$$(\hat{q}_t^t, \hat{b}_1^t, \dots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \hat{b}_{t+2}^t, \dots) = \left(\frac{(2j-5)j^2 + 3^j + F(j)}{3^j 8 j^2}, 0, \dots, 0, \hat{b}_j^2, 0, \dots\right),$$

for $t = 4, 5, \dots$, with $F(y) = \sqrt{(5-2y)^2 y^4 + 3^y 2y^2(6y-11) + 9^y}$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

Proof. At the Cournot-Nash equilibrium, the market power measure of type 1 traders, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{(2j-5)j^2+3^{j+1}-F(j)}{4(3^j-j^2)}$, for each $j \geq 3$. Hence, $\lim_{j\to\infty} \hat{b}_j^1/\bar{\hat{b}}_j = \frac{1}{2}$. ■

In this example, type 1 traders are again asymptotic oligopolists because they placed sufficiently higher bids than other traders in each trading post. Therefore, along the sequence of commodities, the bids made by type 1 traders do not become negligible in comparison to the bids of all others. Heuristically, a trader, with a sufficiently higher initial endowment of commodity money and a sequence of utility coefficients that converges to zero slower than the ones of other traders, can be an asymptotic oligopolist even if the number of traders active in each trading post is not uniformly bounded. In Example 3, indeed, type 1 traders have an higher initial endowment of commodity money than in the previous examples and the sequence $\{\frac{1}{j^2}\}$ converges to zero slower that the sequence $\{\delta^j\}$.

In the last example, we show why the notion of asymptotic oligopolist is defined by using the notion of limit instead of the notion of limit point.

Example 4. Consider an exchange economy in which traders of type 1, 2, s odd, and t even have the following utility functions and initial endowments

$$u^1(x^1) = x_0^1 + x_1^1 + \sum_{j=2}^{\infty} \delta^j \ln x_j^1 \qquad w^1 = \left(\frac{1}{1-\delta}, 1, 0, \dots\right),$$

$$u^2(x^2) = x_0^2 + \delta \ln x_1^2 + x_2^2 + \delta^3 \ln x_3^2 \qquad w^2 = \left(\frac{\delta^1}{1-\delta}, 0, 1, 0, \dots\right),$$

$$u^s(x^s) = x_0^s + x_s^s + \frac{1}{(s+1)^2} \ln x_{s+1}^s \qquad w^s = \left(\frac{1}{(s-1)^2}, 0, \dots, 0, 1, 0, \dots\right),$$

$$u^t(x^t) = x_0^t + x_t^t + \delta^{t+1} \ln x_{t+1}^t \qquad w^t = \left(\frac{\delta^{t-1}}{1-\delta}, 0, \dots, 0, 1, 0, \dots\right).$$

For $k = 2$ and $\delta = 1/3$, the type-symmetric active Cournot-Nash equilibrium of the game Γ associated to the exchange economy is

$$(\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1 \dots, \hat{b}_j^1, \hat{b}_{j+1}^1, \dots) = \left(\frac{1}{12}, \frac{1}{18}, \frac{1}{36}, \dots, \frac{6}{3^j 5 - j^2 + F(j)}, \frac{3}{3^{j+1}4}, \dots\right),$$

$$(\hat{q}_2^2, \hat{b}_1^2, \hat{b}_3^2, \hat{b}_4^2, \dots, \hat{b}_j^2, \dots) = \left(\frac{1}{36}, \frac{1}{6}, \frac{1}{36}, 0, \dots, 0, \dots\right),$$

$$(\hat{q}_s^s, \hat{b}_1^s, \dots, \hat{b}_{s-1}^s, \hat{b}_{s+1}^s, \hat{b}_{s+2}^s, \dots) = \left(\frac{3}{3^s 4}, 0, \dots, 0, \frac{6}{5(s+1)^2 - 3^{s+1} + F(s+1)}, 0, \dots\right),$$

$$(\hat{q}_t^t, \hat{b}_1^t, \dots, \hat{b}_{t-1}^t, \hat{b}_{t+1}^t, \hat{b}_{t+2}^t, \dots) = \left(\frac{t^2 + 3^t + F(t)}{3^t 8 t^2}, 0, \dots, 0, \frac{3}{3^{t+1}4}, 0, \dots\right),$$

for s odd and t even, with $F(y) = \sqrt{y^4 + 3^y 14y^2 + 9^y}$. At this Cournot-Nash equilibrium, type 1 traders are asymptotic oligopolists.

Proof. At the Cournot-Nash equilibrium, the market power measure of trader 1, $\hat{b}_j^1/\bar{\hat{b}}_j$, is equal to $\frac{1}{4}$, for each $j \geq 3$ and odd, and to $\frac{2j^2}{3j^2+F(j)+3^j}$, for each $j \geq 3$ and even. Hence, the sequence of market power measures has two limit points $\frac{1}{4}$ and 0. ∎

In this example, even if the sequence of market power measures has two different limit points and one of them is zero, type 1 traders can be considered asymptotic oligopolists because there is an infinite number of commodities on which they have a non-negligible market power. Moreover, the example shows that the notion of asymptotic oligopolist is very sensitive to the pattern of bids. In fact, if type 1 traders made bids only on even commodities, they would be asymptotic price-takers.

4 Proofs

Following DS, in order to prove the existence of a Cournot-Nash equilibrium, we introduce the perturbed strategic market game Γ^ϵ, the function $x_0^i(x_1^i, x_2^i, \dots)$, and the set $Y^i(s^{-i}, \epsilon)$.[11] The perturbed strategic market game Γ^ϵ is a game defined as Γ with the only exception that the price vector $p(s)$ becomes

$$p_j^\epsilon(s) = \begin{cases} \frac{\bar{b}_j+\epsilon}{\bar{q}_j+\epsilon} & \text{if } \bar{q}_j \neq 0 \\ 0 & \text{if } \bar{q}_j = 0 \end{cases},$$

for each $j \in J \setminus \{0\}$, with $\epsilon > 0$. The interpretation is that some outside agency places a fixed bid of ϵ and a fixed offer of ϵ in each trading post. This does not change the strategy sets of the various traders, but does affect the prices, the commodity bundles, and the payoffs. Consider, without loss of generality, a trader i of type t and fix the strategies s^{-i} for all other traders. Let

$$x_0^i(x_1^i, x_2^i, \dots) = w_0^i + \frac{(\bar{b}_t + \epsilon)(w_t^i - x_t^i)}{\bar{q}_t^i + \epsilon + w_t^i - x_t^i} - \sum_{j \neq 0,t} \frac{(\bar{b}_j^i + \epsilon)x_j^i}{\bar{q}_j + \epsilon - x_j^i}$$

and let

$$Y^i(s^{-i}, \epsilon) = \Big\{(x_1^i, x_2^i, \dots) \in X : x_t^i = w_t^i - q_t^i,\ x_j^i = b_j^i \frac{\bar{q}_j + \epsilon}{\bar{b}_j^i + b_j^i + \epsilon},\ \text{for each } j \in J \setminus \{0, t\},$$
$$\text{for each } s^i \in S^i\Big\}.$$

The function $x_0^i(x_1^i, x_2^i, \dots)$ can be easily obtained by the function $x_0^i(s^i)$ by relabelling the variables. Moreover, it is straightforward to verify that this function is strictly concave since it is a sum of concave and strictly concave functions. In the next Proposition, by following the proof in Appendix A of DS, we prove that $Y^i(s^{-i}, \epsilon)$ is a convex set.

Proposition 1. The set $Y^i(s^{-i}, \epsilon)$ is convex.

Proof. Consider a trader i of type t and fix the strategies s^{-i} for all other traders. Take two commodity bundles $x'^i, x''^i \in Y^i(s^{-i}, \epsilon)$ and consider $\tilde{x}^i = \lambda x'^i + (1-\lambda)x''^i$. We want to show that $\tilde{x}^i \in Y^i(s^{-i}, \epsilon)$. Hence, there must exist a strategy $\tilde{s}^i \in S^i$ such that $x^i(\tilde{s}^i) = \tilde{x}^i$. Let $x'^i = x^i(s'^i)$ and $x''^i = x^i(s''^i)$. Consider first the commodity t. It is straightforward to

[11] DS denotes the set $Y^i(s^{-i}, \epsilon)$ with $D^i(Q, B, \epsilon)$.

verify that $\tilde{x}_t^i = x_t^i(\lambda q_t'^i + (1-\lambda)q_t''^i)$. Consider now a commodity $j \neq t$. The function $x_j^i(\tilde{s}^i)$ is concave in b_j^i, hence

$$\tilde{x}_j^i = \lambda x'^i + (1-\lambda)x''^i = \lambda x^i(b_j'^i) + (1-\lambda)x^i(b_j''^i) \leq x^i(\lambda b_j'^i + (1-\lambda)b_j''^i) \leq x_j^{*i}$$

By the intermediate value theorem and since $x_j^{*i} = 0$ by setting $b_j'^i = 0$ and $b_j''^i = 0$, we may reduce $b_j'^i$ and $b_j''^i$ appropriately to get $\tilde{x}_j^i$, for each $j \in J \setminus \{0, t\}$. ■

In the next lemma, we prove the existence of an active Cournot-Nash equilibrium in the perturbed game.

Lemma 1. Under Assumptions 1, 2, 3, 4, and 5, for each $\epsilon > 0$, a Cournot-Nash equilibrium of the strategic market game Γ^ϵ exists.

Proof. Consider, without loss of generality, a trader i and fix the strategies s^{-i} for all other traders. Let h^i be a function such that $h^i(s^i) = (x_0^i(s^i, s^{-i}), x_1^i(s^i, s^{-i}), \dots)$. Since the game is perturbed, the payoff function π^i is continuous in the product topology. By Tychonoff Theorem (see Theorem 2.61, p. 52 in Aliprantis and Border), S^i is compact. By Weierstrass Theorem (see Corollary 2.35, p. 40 in Aliprantis and Border), there exists a strategy $\hat{s}^i$ that maximizes the payoff function. Hence, we can consider the best response correspondence $\phi^i : S^{-i} \to S^i$. Since S^i is a non-empty and compact Hausdorff space, by Berge Maximum Theorem (see Theorem 17.31, p. 570 in Aliprantis and Border), ϕ^i is an upper hemicontinuous correspondence.

We show now that ϕ^i is a continuous function. Suppose that there are two feasible commodity bundles x^i and x'^i that maximize the utility function. Consider the commodity bundle $x_i''^i = \frac{1}{2}x^i + \frac{1}{2}x'^i$. Since the utility function is concave, $u^i(x''^i) \geq \frac{1}{2}u^i(x^i) + \frac{1}{2}u^i(x'^i) = u^i(x^i)$. Since $x_0^i(x_1^i, x_2^i, \dots)$ is strictly concave and $Y^i(s^{-i}, \epsilon)$ is convex, there exists a $\gamma > 0$ such that $x''^i + \gamma e^0$ is a feasible allocation[12]. But then, since the utility function is strictly increasing in x_0^i, $u^i(x''^i + \gamma e^0) > u^i(x^i)$, a contradiction. Therefore, there must be only one commodity bundle that maximizes the utility function and then ϕ^i is a single valued correspondence. Hence, ϕ^i is a continuous function (see Lemma 17.6, p.559 in Aliprantis and Border).

As we are looking for a fixed point in strategy space S, let's consider $\phi^i : S \to S^i$. Let $\Phi : S \to S$ such that $\Phi(S) = \prod_{i \in I} \phi^i(S)$. The function Φ is continuous since is a product of continuous function (see Theorem 17.28, p. 568 in Aliprantis and Border). By Tychonoff Theorem, S is compact. Moreover, S is a non-empty and convex Hausdorff space. Therefore, by Brouwer-Schauder-Tychonoff Theorem (see Corollary 17.56, p. 583 in Aliprantis and Border), there exists a fixed point $\hat{s}$ of Φ, which is a Cournot-Nash equilibrium of the perturbed game Γ^ϵ. ■

In the next lemma, we prove that, at a Cournot-Nash equilibrium of the perturbed game, a trader makes positive bids on all commodities that he desires.

Lemma 2. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game Γ^ϵ, $\hat{b}_j^i > 0$, for each $j \in L^i$, for each $i \in I$.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. First, suppose that there exists a trader i of type t such that $\hat{b}_l^i = 0$, for an $l \in L^i$, and $\sum_{j \neq 0,t} \hat{b}_j^i < w_0^i$. Consider a strategy s'^i such that $b_l'^i = \hat{b}_l^i + \gamma$, with γ sufficiently small, and all other actions equal to the actions of the original strategy $\hat{s}^i$. Since $\lim_{x_l^i \to 0} \frac{\partial u^i}{\partial x_l^i} = \infty$, $u^i(x^i(s'^i, \hat{s}^{-i})) > u^i(x^i(\hat{s}^i, \hat{s}^{-i}))$, a contradiction. Now, suppose that there exists a trader i of type t such that $\hat{b}_l^i = 0$, for an $l \in L^i$, and $\sum_{j \neq 0,t} \hat{b}_j^i = w_0^i$. Then there is a commodity m such that $\hat{b}_m^i > 0$. Consider a strategy s'^i such that $b_m'^i = \hat{b}_m^i - \gamma$, $b_l'^i = \hat{b}_l^i + \gamma$, with γ sufficiently small, and all other actions equal to the

[12] e^j is an infinite vector in ℓ_∞ whose jth component is 1, and all others are 0.

actions of the original strategy $\hat{s}^i$. Since $\lim_{x_l^i \to 0} \frac{\partial u^i}{\partial x_l^i} = \infty$, $u^i(x^i(s'^i, \hat{s}^{-i})) > u^i(x^i(\hat{s}^i, \hat{s}^{-i}))$, a contradiction. Hence, $\hat{b}_j^i > 0$, for each $j \in L^i$, for each $i \in I$. ■

In the next lemma, we prove that, at a Cournot-Nash equilibria of the perturbed game, $\overline{q}_j$ is positive for all commodities.

Lemma 3. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game Γ^ϵ, $\overline{q}_j > 0$, for each $j \in J \setminus \{0\}$.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i of type t. Then, $\hat{s}^i$ solves the following maximization problem

$$
\begin{aligned}
\max_{s^i} \quad & \pi^i(s^i, \hat{s}^{-i}), \\
\text{subject to} \quad & q_t^i \leq w_t^i, && (i) \\
& \sum_{j \neq 0, t} b_j^i \leq w_0^i, && (ii) \\
& -q_t^i \leq 0, && (iii) \\
& -b_j^i \leq 0 \text{ for each } j \in J \setminus \{0, t\}. && (iv)
\end{aligned}
\tag{2}
$$

We now show that all the hypothesis of the Generalized Kuhn-Tucker Theorem (see Appendix A) are satisfied. The constraints can be written as a function $G : \ell_\infty \to Z$, with $Z \subset \ell_\infty$. Then, Z contains a positive cone P with a non-empty interior and G is Fréchet differentiable. Consider a point $s \in S$. By the definition of regular point (see Appendix A), we must show that there exists an $h \in \ell_\infty$ such that $G(s) + g'(s)\, h < 0$. In matrix form, it becomes

$$
\begin{bmatrix} q_t^i - w_t^i \\ \sum_j b_j^i - w_0^i \\ -q_t^i \\ -b_1^i \\ -b_2^i \\ \cdots \end{bmatrix}
+
\begin{bmatrix} h_t \\ \sum_{j \neq t} h_j \\ -h_t \\ -h_2 \\ -h_3 \\ \cdots \end{bmatrix}
<
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}.
$$

First, suppose that the constraints (i) and (ii) are not binding. Consider a vector h with $h_j = 0$, for all j such that $b_j > 0$, and h_j positive and sufficiently small, for all other j. Hence, s is a regular point. Now, suppose that the constraints (i) and (ii) are binding. Consider a vector h with h_j positive and sufficiently small, for all j such that $b_j = 0$, and h_j negative and sufficiently small, for all other j. Hence, s is a regular point. The case in which only constraint (i) or (ii) is binding can be solved analogously. Hence, all the points in the constrained set are regular. By Lemma 2, the strategy $\hat{s}^i$ is such that $\hat{b}_j^i > 0$, for each $j \in L^i$, and then the commodity bundles $x^i(\hat{s}^i)$ is contained in $X_{L^i > 0}$. Since the utility function is continuously Fréchet differentiable in $X_{L^i > 0}$ and we are considering a perturbed game, the payoff function π^i is continuously Fréchet differentiable in a neighbourhood of $\hat{s}^i$. Hence, all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied and then there exist non-negative multipliers $\hat{\lambda}_1^{i*}$ and $\hat{\mu}_t^{i*}$ such that

$$
\begin{aligned}
& \frac{\partial \pi^i}{\partial q_t^i}(\hat{s}^i, \hat{s}^{-1}) - \hat{\lambda}_1^{i*} + \hat{\mu}_t^{i*} = 0, \\
& \hat{\lambda}_1^{i*}(\hat{q}_t^i - w_t^i) = 0, \\
& \hat{\mu_t}^{i*} \hat{q}_t^i = 0.
\end{aligned}
\tag{3}
$$

Since the payoff function is defined as $\pi^i(s) = u^i(x^i(s))$, equation (3) becomes

$$
\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) \frac{\overline{\hat{b}}_t + \epsilon}{\overline{q}_t^i + \hat{q}_t^i + \epsilon} \left(1 - \frac{\hat{q}_t^i}{\overline{q}_t^i + \hat{q}_t^i + \epsilon} \right) - \frac{\partial u^i}{\partial x_t^i}(x^i(\hat{s})) - \hat{\lambda}_1^{i*} + \hat{\mu}_t^{i*} = 0. \tag{4}
$$

Suppose that $\overline{q}_t = 0$. Then, $\hat{q}^i_t = 0$, for all traders of type t, and $\hat{\lambda}^{i*}_1 = 0$. Then, the equation above becomes

$$\frac{\partial u^i}{\partial x^i_0}(x^i(\hat{s}))\frac{\overline{\hat{b}}_t + \epsilon}{\epsilon} - \frac{\partial u^i}{\partial x^i_t}(x^i(\hat{s})) + \hat{\mu}^{i*}_t = 0.$$

Since $\frac{\overline{\hat{b}}_t+\epsilon}{\epsilon} > 1$ and $\frac{\partial u^i}{\partial x^i_0}(x^i) \geq \frac{\partial u^i}{\partial x^i_t}(x^i)$, for each $x^i \in X$, by Assumption 3, the left hand side of the equation is greater than zero, a contradiction. Hence, $\overline{q}_j > 0$, for all $j \in J \setminus \{0\}$. ■

In the next lemma, we prove that all prices lie between two positive constants independent from ϵ. Since the number of commodities is infinite, we could not apply the analogous lemma of DS.

Lemma 4. At a Cournot-Nash equilibrium $\hat{s}$ of the perturbed game Γ^ϵ, there exist two positive constants, independent from ϵ, C_j and D_j such that

$$C_j < p^\epsilon_j(\hat{s}) < D_j,$$

for each $j \in J \setminus \{0\}$. Moreover, C_j is uniformly bounded away from zero and D_j is uniformly bounded from above.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Without loss of generality, let $j = l$. We establish first the existence of C_l. By Lemma 3, there exists a trader i of type l such that $\hat{q}^i_l > 0$. Then, a decrease γ in i's offer of commodity l is feasible, with $0 < \gamma \leq \hat{q}^i_l$, and has the following incremental effects on the final holding of trader i.

$$\begin{aligned}
x^i_0(\hat{s}(\gamma)) - x^i_0(\hat{s}) &= (\hat{q}^i_l - \gamma)\frac{\overline{\hat{b}}_l + \epsilon}{\overline{\hat{q}}_l + \epsilon - \gamma} - \hat{q}^i_l\frac{\overline{\hat{b}}_l + \epsilon}{\overline{\hat{q}}_l + \epsilon}, \\
&= \frac{\overline{\hat{b}}_l + \epsilon}{\overline{\hat{q}}_l + \epsilon}\left((q^i_l - \gamma)\frac{\overline{\hat{q}}_l + \epsilon}{\overline{\hat{q}}_l + \epsilon - \gamma} - \hat{q}^i_l\right) \geq -p^\epsilon_l(\hat{s})\gamma, \\
x^i_j(\hat{s}(\gamma)) - x^i_j(\hat{s}) &= 0, \text{ for } j \in J \setminus \{0, l\}, \\
x^i_l(\hat{s}(\gamma)) - x^i_l(\hat{s}) &= \gamma.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\overline{\hat{q}}_l + \epsilon > \overline{\hat{q}}_l + \epsilon - \gamma$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) - p^\epsilon_l(\hat{s})\gamma e^0 + \gamma e^l.$$

By Lemma 2, $x^i(\hat{s}) \in X_{L^i>0}$, and by using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u(x^i(\hat{s}(\gamma))) - u(x^i(\hat{s})) \geq -\frac{\partial u^i}{\partial x^i_0}(x^i(\hat{s}))p^\epsilon_l(\hat{s})\gamma + \frac{\partial u^i}{\partial x^i_l}(x^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p^\epsilon_l(\hat{s}) > \frac{\partial u^i}{\partial x^i_l}(x^i(\hat{s}))\Big/\frac{\partial u^i}{\partial x^i_0}(x^i(\hat{s})) = C_l.$$

By Assumption 3, $C_l \geq \frac{\lambda}{f}$. Then, $C_j \geq \frac{\lambda}{f}$, for each $j \in J \setminus \{0\}$. Now, we establish the existence of D_l. Since there are at least two traders for each type, we consider a trader i of type l such that $q^i_l \leq \frac{\overline{\hat{q}}_l}{2}$. We need to consider two cases. First, suppose that $\hat{q}^i_l < w^i_l$. Then, an increase

$0 < \gamma < \min\{w^i_l - \hat{q}^i_l, \epsilon\}$ in i's offer of l is feasible and has the following incremental effects on the final holding of trader i

$$\begin{aligned}
x^i_0(\hat{s}(\gamma)) - x^i_0(\hat{s}) &= (\hat{q}^i_l + \gamma)\frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon + \gamma} - \hat{q}^i_l \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon}, \\
&= \frac{\bar{\hat{b}}_l + \epsilon}{\bar{\hat{q}}_l + \epsilon} \frac{\bar{\hat{q}}^i_l + \epsilon}{\bar{\hat{q}}^i_l + \hat{q}^i_l + \epsilon + \gamma}\gamma \geq \frac{1}{3} p^\epsilon_l(\hat{s})\gamma, \\
x^i_j(\hat{s}(\gamma)) - x^i_j(\hat{s}) &= 0, \text{ for } j \in J \setminus \{0, l\}, \\
x^i_l(\hat{s}(\gamma)) - x^i_l(\hat{s}) &= -\gamma.
\end{aligned}$$

The inequality in the preceding array follows from the fact that $\hat{q}^i_l \leq \bar{\hat{q}}^i_l + \epsilon$ and $\gamma \leq \bar{\hat{q}}^i_l + \epsilon$. Then, we obtain the following vector inequality

$$x^i(\hat{s}(\gamma)) \geq x^i(\hat{s}) + \frac{1}{3} p^\epsilon_l(\hat{s})\gamma e^0 - \gamma e^l.$$

By Lemma 2, $x^i(\hat{s}) \in X_{L^i > 0}$, and by using a linear approximation of the utility function around the point $x^i(\hat{s})$, we obtain

$$u(x^i(\hat{s}(\gamma))) - u(x^i(\hat{s})) \geq \frac{\partial u^i}{\partial x^i_0}(x^i(\hat{s}))\frac{1}{3} p^\epsilon_l(\hat{s})\gamma - \frac{\partial u^i}{\partial x^i_l}(x^i(\hat{s}))\gamma + O(\gamma^2).$$

Since $x^i(\hat{s})$ is an optimum point, the left hand side of the inequality is negative and then

$$p^\epsilon_l(\hat{s}) < 3\left(\frac{\partial u^i}{\partial x^i_l}(x^i(\hat{s})) \Big/ \frac{\partial u^i}{\partial x^i_0}(x^i(\hat{s}))\right) = D^1_l.$$

By Assumption 3, $D^1_l \leq 3\frac{f}{\lambda}$. Now, suppose that $\hat{q}^i_l = w^i_l$. Then,

$$p^\epsilon_l(\hat{s}) < \frac{ke}{w^i_l} = D^2_l.$$

By Assumption 1, $D^2_l \leq \frac{ke}{\sigma}$. Finally, we choose D_l such that $D_l = \max\{D^1_l, D^2_l\}$. Since D^1_j and D^2_j are uniformly bounded from above, D_j is uniformly bounded from above, for each $j \in J \setminus \{0\}$. ■

In the next lemma, we prove that, at an active Cournot-Nash equilibrium, there exists a positive lower bound, independent from ϵ, for each bid made by a trader on a desired commodity.

Lemma 5. At an active Cournot-Nash equilibrium $\hat{s}$ of the perturbed game Γ^ϵ, there exists a positive constant $\underline{b}^i_j$, independent of ϵ, such that

$$0 < \underline{b}^i_j \leq \hat{b}^i_j,$$

for each $j \in L^i$, for each $i \in I$.

Proof. Let $\hat{s}$ be a Cournot-Nash equilibrium of the perturbed game. Consider, without loss of generality, a trader i of type t. Then, $\hat{s}^i$ solves the maximization problem in (2). As in the proof of Lemma 3, all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied and then there exist non-negative multipliers $\hat{\lambda}^{i*}_2$ and $\hat{\mu}^{i*}_j$, for $j \in J \setminus \{0, t\}$, such that

$$\begin{aligned}
&\frac{\partial \pi^i}{\partial b^i_j}(\hat{s}^i, \hat{s}^{-1}) - \hat{\lambda}^{i*}_2 + \hat{\mu}^{i*}_j = 0, \text{ for each } j \in J \setminus \{0, t\}, \qquad (5) \\
&\hat{\lambda}^{i*}_2\Big(\sum_{j \neq 0, t} \hat{b}^i_j - w^i_0\Big) = 0, \\
&\hat{\mu}^{i*}_j \hat{b}^i_j = 0, \text{ for each } j \in J \setminus \{0, t\}.
\end{aligned}$$

Consider, without loss of generality, a commodity $l \in L^i$. Since the payoff function is defined as $\pi^i(s) = u^i(x^i(s))$, equation (5) becomes

$$-\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{\bar{\hat{q}}_l + \epsilon}{\bar{\hat{b}}_l^i + b_l^i + \epsilon}\left(1 - \frac{b_l^i}{\bar{\hat{b}}_l^i + b_l^i + \epsilon}\right) - \hat{\lambda}_2^{i*} + \hat{\mu}_j^{i*} = 0. \tag{6}$$

From the equation above and by Lemmas 2 and 4, we derive the following inequality

$$-\frac{\partial u^i}{\partial x_0^i}(x^i(\hat{s})) + \frac{\partial u^i}{\partial x_l^i}(x^i(\hat{s}))\frac{1}{D_l}\left(\frac{\bar{\hat{b}}_l^i + \epsilon}{\bar{\hat{b}}_l^i + b_l^i + \epsilon}\right) - \hat{\lambda}_2^{i*} \leq 0.$$

Suppose that $b_l^i \to 0$. Then, $\lim_{b_l^i \to 0} \frac{\bar{\hat{b}}_l^i + \epsilon}{\bar{\hat{b}}_l^i + b_l^i + \epsilon} = 1$ and $\lim_{b_l^i \to 0} \frac{\partial u^i}{\partial x_l^i} = \infty$, as $l \in L^i$. But then, since $\frac{\partial u^i}{\partial x_0^i}$ and D_l are bounded from above, the left hand side of the inequality is positive, a contradiction. Hence, there exists a positive lower bound, $\underline{b}_j^i$, independent of ϵ, at which the inequality is satisfied, for each $j \in L^i$, for each $i \in I$. ■

We now prove Theorem 1.

Proof of Theorem 1. Consider a sequence of $\{{}^g\epsilon\}_{g=1}^{\infty}$ converging to 0. By Lemma 1, in each perturbed game there exists a Cournot-Nash equilibrium. Then, we can consider the sequence of Cournot-Nash equilibria $\{{}^g\hat{s}\}_{g=1}^{\infty}$ associated to the sequence of ϵ. As proved before, S is compact and, by Lemma 4, $p^{\epsilon}({}^g\hat{s}) \in \prod_{j\neq 0}[C_j, D_j]$ with C_j uniformly bounded away from zero and D_j uniformly bounded from above, for each $j \in J \setminus \{0\}$. By Tychonoff Theorem, $\prod_{j\neq 0}[C_j, D_j]$ is compact. Then, we can pick a subsequence of $\{{}^g\hat{s}\}_{g=1}^{\infty}$ that converge to v such that $v \in S$ and $p(v) \in \prod_{j\neq 0}[C_j, D_j]$. Therefore, v is a point of continuity of the payoff functions and then it is a Cournot-Nash equilibrium, i.e., $\hat{v}$. It remains to prove that $\hat{v}$ is an active Cournot-Nash equilibrium. By Assumption 4 and Lemma 5, for each commodity $j \in J \setminus \{0\}$, $\bar{b}_j \in [\underline{b}_j^i, e]$, for any ϵ, for a trader i such that $j \in L^i$. Suppose that one trading post is inactive. Then, there exists a commodity l such that $\bar{q}_l = 0$. But then, $p_l(\hat{v}) \notin [C_l, D_l]$, a contradiction. Therefore, $\bar{q}_j > 0$, for each $j \in J \setminus \{0\}$, and then $\hat{v}$ is an active Cournot-Nash equilibrium. ■

In the next part of the section, we prove Theorem 2. Before to do so, we define the marginal price vector[13] and we prove the analogous of Lemma 4 of DS for our setting with infinitely many commodities.

Definition 5. Consider an active Cournot-Nash equilibrium $\hat{s}$ of Γ. The marginal price vector for a trader i of type t, $\overline{p}_t^i(\hat{s})$, is such that

$$\overline{p}_t^i(\hat{s}) = p_t(\hat{s})\left(1 - \frac{\hat{q}_t^i}{\bar{\hat{q}}_t}\right) \text{ and } \overline{p}_j^i(\hat{s}) = p_j(\hat{s})\left(1 + \frac{\hat{b}_j^i}{\bar{\hat{b}}_j^i}\right), \text{ for all } j \in J \setminus \{0, t\}.$$

Lemma 6. At an interior type-symmetric active Cournot-Nash equilibrium ${}_k\hat{s}$ of the game ${}_k\Gamma$, a trader i of type t maximizes his payoff at the fixed marginal price vector ${}_k\overline{p}^i({}_k\hat{s})$, for each $i \in I$.

Proof. Let ${}_k\hat{s}$ be an interior type-symmetric active Cournot-Nash equilibrium of the game ${}_k\Gamma$. Consider, without loss of generality, a trader i of type t and the following maximization

[13] This definition is equivalent to the one of Okuno et al. (1980) only if the Cournot-Nash equilibrium is such that $0 < \hat{q}_t^i < w_t^i$, $\hat{b}_j^i > 0$, for $j \in J \setminus \{0, t\}$, and $\sum_{j\neq 0,t} \hat{b}_j^i < w_0^i$.

problem

$$
\begin{aligned}
\max_{{}_k s^i} \quad & \pi^i({}_k s^i, \overline{p}^i({}_k\hat{s})), & \\
\text{subject to} \quad & q_t^i \leq w_t^i, & (i) \\
& \sum_{j\neq 0,t} b_j^i \leq w_0^i, & (ii) \\
& -q_t^i \leq 0, & (iii) \\
& -b_j^i \leq 0, \ \text{for each } j \in J\setminus\{0,t\}, & (iv)
\end{aligned}
\tag{7}
$$

with $\pi^i({}_k s^i, \overline{p}^i({}_k\hat{s}))$ a payoff function at which the price vector is fixed and equal to $\overline{p}^i({}_k\hat{s})$. Let $x^i({}_k s^i, \overline{p}^i({}_k\hat{s}))$ denote the commodity bundle of trader i when he plays ${}_k s^i$ and the price vector is fixed and equal to $\overline{p}^i({}_k\hat{s})$. As in the proof of Lemma 3, all the hypothesis of the Generalized Kuhn-Tucker Theorem are satisfied and then, if a ${}_k s^i$ solves the maximization problem, there exist non-negative multipliers λ_1^{i*}, λ_2^{i*} and μ_j^{i*}, for $j \in J\setminus\{0\}$, such that

$$
\frac{\partial u^i}{\partial x_0^i}(x^i({}_k s^i, \overline{p}^i({}_k\hat{s})))\overline{p}_t^i({}_k\hat{s}) - \frac{\partial u^i}{\partial x_t^i}(x^i({}_k s^i, \overline{p}^i({}_k\hat{s}))) - \lambda_1^{i*} + \mu_t^{i*} = 0, \tag{8}
$$

$$
\lambda_1^{i*}(\hat{q}_t^i - w_t^i) = 0,
$$

$$
\mu_t^{i*}\hat{q}_t^i = 0,
$$

$$
-\frac{\partial u^i}{\partial x_0^i}(x^i({}_k s^i, \overline{p}^i({}_k\hat{s}))) + \frac{\partial u^i}{\partial x_j^i}(x^i({}_k s^i, \overline{p}^i({}_k\hat{s})))\frac{1}{\overline{p}_j^i({}_k\hat{s})} - \lambda_2^{i*} + \mu_j^{i*} = 0, \tag{9}
$$

$$
\lambda_2^{i*}(\sum_{j\neq 0,t}\hat{b}_j^i - w_0^i) = 0,
$$

$$
\mu_j^{i*}\hat{b}_j^i = 0,
$$

for each $j \in J\setminus\{0,t\}$. By using the definition of marginal price vector, it is straighfoward to verify that equations (4) and (6) become (8) and (9) respectively. But then, ${}_k\hat{s}^i$, $\hat{\lambda}_1^{i*}$, $\hat{\lambda}_2^{i*}$, and $\hat{\mu}_j^{i*}$, for $j \in J\setminus\{0\}$, satisfy all the first order conditions associated to the maximization problem (7). Since the utility function is concave and prices are fixed, the payoff function $\pi^i({}_k s^i, \overline{p}^i({}_k\hat{s}))$ is concave. Hence ${}_k\hat{s}^i$ is optimal for the problem (7).[14] ■

We now prove Theorem 2.

Proof of Theorem 2. Let $\{{}_k\Gamma\}_{k=2}^{\infty}$ be a sequence of games ${}_k\Gamma$. Assume that there exists a sequence of interior type-symmetric active Cournot-Nash equilibria $\{{}_k\hat{s}\}_{k=2}^{\infty}$ such that the sequences $\{{}_k\tilde{s}\}_{k=2}^{\infty}$ and $\{p({}_k\tilde{s})\}_{k=2}^{\infty}$ converge to $\tilde{v}$ and to $p(\tilde{v})$ respectively. Consider, without loss of generality, a trader i of type t. By Lemma 6, ${}_k\hat{s}^i$ solves the maximization problem (7), for any k, and since ${}_k\hat{s}^i$ is an interior type symmetric active Cournot-Nash equilibrium, the constraints (ii) and (iii) are not binding. Let $\overline{p}_0({}_k\hat{s}) = 1$, for any k.[15] It is straightfoward to verify that $x^i({}_k\hat{s}^i, \overline{p}^i({}_k\hat{s}))$ belongs to the budget set at price $\overline{p}^i({}_k\hat{s})$, $B^i(\overline{p}^i({}_k\hat{s}))$, for any k. Now, suppose that there exists a commodity bundle $x'^i \in B^i(\overline{p}^i({}_k\hat{s}))$ such that $u^i(x'^i) > u^i(x^i({}_k\hat{s}^i, \overline{p}^i({}_k\hat{s})))$. Since the utility function is non decreasing, $x_j'^i > x_j^i({}_k\hat{s}^i, \overline{p}^i({}_k\hat{s}))$, for at least one commodity j. But since $\sum_{j\neq 0,t}\hat{b}_j^i < w_0^i$ and $-\hat{q}_t^i < 0$, there exists a feasible strategy ${}_k s'^i \in S^i$ such that $x_j^i({}_k s'^i, \overline{p}^i({}_k\hat{s})) = x_j'^i$, a contradiction. Hence, the commodity bundle $x^i({}_k\hat{s}^i, \overline{p}^i({}_k\hat{s}))$ maximizes the utility function on $B^i(\overline{p}^i({}_k\hat{s}))$, for each trader $i \in I$, for any k. Now, consider the sequence $\{\overline{p}^t({}_k\tilde{s})\}_{k=2}^{\infty}$, for a representative trader of type t. By the assumptions of the Theorem and by the definition of marginal price vector, $\lim_{k\to\infty}\overline{p}_j^t({}_k\tilde{s}) =$

[14] This conclusion can be also formally obtained by Theorem 2 of Section 8.5 and Lemma 1 of Section 8.7 in Luenberger (1969).

[15] With a slight abuse of notation, $\overline{p}({}_k\hat{s})$ denotes also a marginal price vector in which the first element is $\overline{p}_0({}_k\hat{s})$.

$p_j(\tilde{v})$, for each $j \in J \setminus \{0\}$, for $t = 1, 2, \ldots$. Since $\hat{s}$ is a type symmetric Cournot-Nash equilibrium, D_j^2 in Lemma 4 becomes $\frac{e}{\sigma}$. Then, C_j and D_j are independent from k, for each $j \in J \setminus \{0\}$. But then, by Lemma 4, $p(\tilde{v}) \in \prod_{j \neq 0}[C_j, D_j]$. Therefore, $\tilde{v}$ is a point of continuity of the payoff function and then the commodity bundle $x^t(\tilde{v})$ is optimal on $B^i(p(\tilde{v}))$, for $t = 1, 2, \ldots$. Hence $(p(\tilde{v}), {}_k\mathbf{x}(\tilde{v}))$ is a Walras equilibrium for the exchange economy associated to ${}_k\Gamma$, for any k. ■

5 Conclusion

In this paper, we have extended the non-cooperative analysis of oligopoly to exchange economies with infinitely many commodities. For a strategic market game with a countable infinity of commodities, we have proved the existence of an active Cournot-Nash equilibrium and the converge to the Walras equilibrium when traders are replicated. Moreover, we have examined, via a number of examples, how the notion of oligopoly changes in our setting by distinguishing between asymptotic oligopolists, the traders whose market is uniformly bounded away from zero on an infinite subset of commodities, and asymptotic price-takers, the ones whose market power converges to zero over the set of commodities.

In the first example, we have shown that some traders are asymptotic oligopololists because, heuristically, traders have similar preferences and only few of them are active in each trading post. In the second example, we have shown a case in which the number of trader active in each trading post is non uniformly bounded and for this reason all traders are asymptotic price-takers. In the third example, we have shown that, even if the number of traders active in each trading post is not uniformly bounded, some traders with an higher initial endowment of commodity money and with particular preferences can be asymptotic oligopolists. All the examples have shown that whether or not a trader is an asymptotic oligopolist in equilibrium is sentitive to the initial endowment of commodity money and to the "size of the market" in each trading post, which depends on preferences throughout traders.

In the last example, we have shown why we have used the notion of limit to define an asymptotic oligopolist instead of the notion of limit point. Furthermore, by using the limit, asymptotic price-takers are approximate price-taker on the tail of the sequence of commodities. Hence, heuristically, they have market power locally but they are globally negligibile, i.e., they have market power on a finite set of commodies but they are approximate price taker on an infinite set of commodities at the tail of the sequence of commodities. In future research, we intend to develop this approach to study some key features of monopolistic competition in strategic market games. A similar idea was devoloped by Gretsky and Ostroy (1985) and Ostroy and Zame (1994) by studying the core of economies with a continuum of commodities and traders.

As already remarked, by using the Hotelling line, our setting can be interpreted as a model of oligopoly with differentiated commodities and thus useful for the study of antitrust policy issues. We intend to pursue this point in future research.

A Mathematical appendix

In this appendix, we describe the mathematical notions that we use in the paper. The definitions and the theorems are based on Luenberger (1969). The page number in brackets refers to Luenberger (1969).

Definition (ℓ_∞ spaces)**.** The space ℓ_∞ consists of bounded sequences. The norm of an element $x = \{\xi_i\}$ in ℓ_∞ is defined as $\|x\|_\infty = \sup_i |x_i|$ (p. 29).

As already remarked, by assuming that the space ℓ_∞ in endowed with the product topology, we impose on ℓ_∞ the norm $\|x\|_\infty = \sup_i \|a_i x_i\|$ such that $\{a_i\}$ is a sequence of real number converging to zero (see Brown and Lewis (1981)).

Definition (Transformation T)**.** Let X and Y be linear vector spaces and let D be a subset of X. A rule which associates with every element $x \in D$ an element $y \in Y$ is said to be a transformation from X to Y with domain D. If y correspond to x under T, we write $y = T(x)$ (p. 27).

Definition (Fréchet differentiable)**.** Let T be a transformation defined on an open domain D in a normed space X and having range in a normed space Y. If for fixed $x \in D$ and each $h \in X$ there exists $\delta T(x; h) \in Y$ which is linear and continuous with respect to h such that

$$\lim_{\|h\| \to 0} \frac{\|T(x+h) - T(x) - \delta T(x; h)\|}{\|h\|} = 0$$

then T is said to be Fréchet differentiable at x and $\delta T(x; h)$ is said to be the Fréchet differential of T at x with incremental h (p. 172).

Definition (Continuously Fréchet differentiable)**.** The Fréchet differential of T at x with incremental h can be written as $T'(x)h$. If the correspondence $x \to T'(x)$ is continuous at the point x_0, we say that the Fréchet derivative of T is continuous at x_0. If the derivative of T is continuous on some open sphere S, we say that T is continuously Fréchet differentiable on S (p. 175).

Definition (Normed dual)**.** Let X be a normed linear vector space. The space of all bounded linear function on X is called the normed dual of X and is denoted by X^* (p. 106).

Luenberger states the Regular Point definition (p. 248) and the Generalized Kuhn-Tucker Theorem (p. 249-250) for vector spaces. Since we deal with normed spaces, we state them for this particular spaces (Example 1, p. 250). Moreover, since it is not crucial for the proof, we require only that f and G are Fréchet differentiable in a neighbourhood of the maximum x_0 and not on all the domain (e.g. Maurer and Zowe (1979)).

Definition (Regular Point)**.** Let X be a normed vector space and let Z be a normed vector space with a closed positive cone P having non-empty interior. Let G be a mapping $G : X \to Z$ which is Fréchet differentiable. A point $x_0 \in X$ is said to be a regular point of the inequality $G(x) \leq 0$ if $G(x_0) \leq 0$ and there is an $h \in X$ such that $G(x_0) + g'(x_0) \cdot h < 0$.

Theorem (Generalized Kuhn-Tucker Theorem)**.** Let X be a normed vector space and Z be a normed vector space having a closed positive cone P. Assume that P contains an interior point. Let f be a real-valued function on X and G a mapping from X into Z. Suppose x_0 maximizes f subject to $G(x) \leq 0$ and that x_0 is regular point of the inequality $G(x) \leq 0$. Moreover suppose that f and G are Fréchet differentiable in a neighbourhood of x_0. Then there is a $z_0^* \in Z^*$, $z_0^* \geq 0$ such that

$$f'(x_0) + z_0^* G'(x_0) = 0$$
$$z_0^* \cdot G(x_0) = 0$$

B The strategic market game introduced by DS

In this appendix, we show the relationship between the game Γ and the strategic market game analysed by DS. The main difference is that in their game traders are allowed to sell and buy the same commodity. We first introduce the original game analysed by DS. Let's call this game $\underline{\Gamma}$. The strategy set of trader i is

$$\underline{S}^i = \Big\{ \underline{s}^i = (\underline{q}_1^i, \underline{b}_1^i, \underline{q}_2^i, \underline{b}_2^i, \ldots, \underline{q}_j^i, \underline{b}_j^i, \ldots) : 0 \leq \underline{q}_j^i \leq w_j^i,\ \underline{b}_j^i \geq 0,\ \text{for each } j \in J \setminus \{0\}, \text{ and } \sum_{j=1}^{\infty} \underline{b}_j^i \leq w_0^i \Big\}.$$

For each $\underline{s} \in \underline{S}$, the price vector $p(\underline{s})$ is such that

$$p_j(\underline{s}) = \begin{cases} \frac{\bar{b}_j}{\bar{q}_j} & \text{if } \bar{q}_j \neq 0 \\ 0 & \text{if } \bar{q}_j = 0 \end{cases},$$

for each $j \in J \setminus \{0\}$, with $\bar{q}_j = \sum_{i \in I} \underline{q}_j^i$ and $\bar{b}_j = \sum_{i \in I} \underline{b}_j^i$. For each $s \in S$, the commodity bundle $x^i(\underline{s})$ of a trader i is such that

$$x_0^i(\underline{s}) = w_0^i - \sum_{j=1}^{\infty} \underline{b}_j^i + \sum_{j=1}^{\infty} \underline{q}_j^i p_j(\underline{s}),$$

$$x_j^i(\underline{s}) = \begin{cases} w_j^i - \underline{q}_j^i + \frac{\underline{b}_j^i}{p_j(\underline{s})} & \text{if } p_j(\underline{s}) \neq 0 \\ 0 & \text{if } p_j(\underline{s}) = 0 \end{cases}, \text{ for each } j \in J \setminus \{0\}.$$

The payoff function of trader i is such that $\underline{\pi}^i(\underline{s}) = u^i(x^i(\underline{s}))$.

Proposition 2. Consider an exchange economy having a multilateral oligopoly structure. An allocation $\mathbf{x}$ is attainable at a Cournot-Nash equilibrium $\hat{s}$ of the game Γ if and only if the same allocation $\mathbf{x}$ is attainable at a Cournot-Nash equilibrium $\hat{\underline{s}}$ of the game $\underline{\Gamma}$.

Proof. Let $\mathcal{E}$ be an exchange economy having a multilateral oligopoly structure. Let $\hat{s}$ be Cournot-Nash equilibrium of the game Γ. Let $\underline{s}$ be a strategy profile such that, for a trader i of type t, $(\underline{q}_1^i, \underline{b}_1^i, \underline{q}_2^i, \underline{b}_2^i, \ldots, \underline{q}_t^i, \underline{b}_t^i, \ldots) = (0, \hat{b}_1^i, 0, \hat{b}_2^i, \ldots, \hat{q}_t^i, 0, \ldots)$, for each $i \in I$. It is straightforward to verify that $\mathbf{x}(\hat{s})$ and $\mathbf{x}(\underline{s})$ are equal. Suppose that $\underline{s}$ is not a Cournot-Nash equilibrium. Then, there exists a trader i of type t that can increase his payoff by playing a strategy $\underline{s}'^i$. The only action that can increase the payoff of the trader is to increase $\underline{b}_t'^i$, because all other feasible deviations are available also in the game Γ. Then, $x_t^i(\underline{s}'^i, \underline{s}^{-i}) > x_t^i(\hat{s})$. But, by decreasing $\hat{q}_t^i$, the commodity bundle $x_t^i(\underline{s}'^i, \underline{s}^{-i})$ is attainable also in the original game, a contradiction. Hence, $\underline{s}$ is a Cournot-Nash equilibrium of $\underline{\Gamma}$. Let $\hat{\underline{s}}$ be Cournot-Nash equilibrium of the game $\underline{\Gamma}$. Let s be a strategy profile such that, for a trader i of type t, $(q_t^t, b_1^i, b_2^i, \ldots, b_{t-1}^i, b_{t+1}^i, \ldots) = (\hat{\underline{q}}_t^i - \frac{\hat{\underline{b}}_t^i}{p_t(\hat{\underline{s}})}, \hat{\underline{b}}_1^i, \hat{\underline{b}}_2^i, \ldots, \hat{b}_{t-1}^i, \hat{b}_{t+1}^i, \ldots)$, for each $i \in I$. It is straightforward to verify that $\mathbf{x}(\hat{\underline{s}})$ and $\mathbf{x}(s)$ are equal. Suppose that s is not a Cournot-Nash equilibrium. Then, there exists a trader i of type t that can increase his payoff by playing a strategy s'^i. But, any possible deviation in Γ is available in $\underline{\Gamma}$, a contradiction. Hence, s is a Cournot-Nash equilibrium of Γ. ■

C Marginal prices and average prices

Okuno et al. (1980) used a different way to measure the market power of traders. They introduce the notions of marginal price and average price and they show that when the two prices are equal traders have no market power. At a Cournot-Nash equilibrium $\hat{s}$, the average price vector is equal to $p(\hat{s})$ and the marginal price vector is defined in Definition 5. The next proposition shows the relationship between our market power measure and the notions of average price and marginal price.

Proposition 3. Consider an active Cournot-Nash equilibrium $\hat{s}$ of Γ. For a trader i of type t, the marginal price vector is equal to the average price vector if and only if $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, for each $j \in J \setminus \{0, t\}$.

Proof. Let $\hat{s}$ be an active Cournot-Nash equilibrium of Γ. Consider a trader i of type t. First, suppose that $\overline{p}^i(\hat{s}) = p(\hat{s})$. This implies that $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j^i = 0$, for each $j \in J \setminus \{0, t\}$. But then, also $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, for each $j \in J \setminus \{0, t\}$. Now, suppose that $\hat{q}_t^i/\bar{\hat{q}}_t = 0$ and $\hat{b}_j^i/\bar{\hat{b}}_j = 0$, $j \in J \setminus \{0, t\}$. This implies that $\hat{b}_j^i = 0$ and then $\hat{b}_j^i/\bar{\hat{b}}_j^i = 0$. Hence $\overline{p}^i(\hat{s}) = p(\hat{s})$. ■

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