Nondictatorial Arrovian Social Welfare Functions: An Integer Programming Approach*

Francesca Busetto[†], Giulio Codognato[‡], Simone Tonin[§]

Abstract

In the line opened by Kalai and Muller (1977), we explore new conditions on preference domains which make it possible to avoid Arrow's impossibility result. In our main theorem, we provide a complete characterization of the domains admitting nondictatorial Arrovian social welfare functions with ties (i.e. including indifference in the range) by introducing a notion of strict decomposability. In the proof, we use integer programming tools, following an approach first applied to social choice theory by Sethuraman, Teo and Vohra ((2003), (2006)). In order to obtain a representation of Arrovian social welfare functions whose range can include indifference, we generalize Sethuraman et al.'s work and specify integer programs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$: indeed, we show that there exists a one-to-one correspondence between the solutions of an integer program defined on this set and the set of all Arrovian social welfare functions - without restrictions on the range.

Journal of Economic Literature Classification Number: D71.

---

*This paper has been written to honor Nick Baigent and his distinguished contributions to social choice theory. We would like to thank an anonymous referee for his comments and suggestions. Francesca Busetto and Giulio Codognato gratefully acknowledge financial support from MIUR (PRIN 20103S5RN3).

[†]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, Via Tomadini 30, 33100 Udine, Italy.

[‡]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, Via Tomadini 30, 33100 Udine, Italy, and EconomiX, Université de Paris Ouest Nanterre la Défense, 200 Avenue de la République, 92001 Nanterre Cedex, France.

[§]Adam Smith Business School, University of Glasgow, Main Building, Glasgow, G128QQ, United Kingdom.

1 Introduction

Arrow (1963) established his celebrated impossibility theorem for Arrovian Social Welfare Functions (ASWFs) - that is social welfare functions satisfying the hypotheses of Pareto optimality and independence of irrelevant alternatives - defining them on the unrestricted domain of preference orderings. As is well known, this result holds also for ASWFs defined on the domain of all antisymmetric preference orderings. Kalai and Muller (1977) dealt with the problem of introducing restrictions on this latter domain of individual preferences in order to overcome Arrow's impossibility result.[1] They gave the first complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs "without ties" - that is ASWFs which do not admit indifference between distinct alternatives in their range. They did this by means of two theorems: in their Theorem 1, they showed that there exists a n-person nondictatorial ASWF for a given domain of antisymmetric preference orderings if and only if there exists a 2-person nondictatorial ASWF for the same domain; in their Theorem 2, they gave the domain characterization, by introducing the concept of decomposability.

In this paper, we proceed along the way opened by Kalai and Muller, and explore new conditions on preference domains which allow for the existence of nondictatorial ASWFs. In fact, Kalai and Muller's Theorem 2 provides a complete characterization of the domains of antisymmetric preference orderings admitting nondictatorial ASWFs without ties and of those admitting dictatorial ASWFs without ties. The problem of characterizing the domains of antisymmetric preference orderings admitting nondictatorial ASWFs "with ties" - that is ASWFs which admit indifference between distinct alternatives in their range - has so far been left open. Here, we overcome this problem: in our main theorem, we provide a complete characterization of these domains by introducing the notion of strict decomposability.

We develop our analysis on nondictatorial ASWFs by using the tools of integer programming, first applied to the traditional field of social choice theory by Sethuraman, Teo, and Vohra ((2003), (2006)). As remarked by these authors, integer programming is a powerful analytical tool, which makes it possible to derive, in a systematic and simple way, many of the already known theorems on ASWFs, and to prove new results.

In particular, Sethuraman et al. developed Integer Programs (IPs) in

[1] Maskin (1979) independently investigated the same issue.

which variables assume values only in the set $\{0, 1\}$. Binary IPs of this kind are suitable to be used as an auxiliary tool to represent ASWFs without ties: a fundamental theorem in Sethuraman et al. (2003) establishes a one-to-one correspondence, on domains of antisymmetric preference orderings, between the set of feasible solutions of their main binary IP and the set of ASWFs without ties. In both papers mentioned above, Sethuraman et al. used binary integer programming to analyze, among other issues, neutral and anonymous ASWFs. Moreover, in the 2003 paper, they opened the way to a reconsideration, in terms of integer programming, of the work by Kalai and Muller (1977). In particular, they provided a simplified version of Kalai and Muller's Theorem 1 by using a binary IP.

In this paper, we extend Sethuraman et al's approach in order to obtain a general representation of ASWFs, without restrictions on the range. To this end, we specify IPs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$. We call these programs "ternary IPs," with some abuse with respect to the current specialized literature.[2] Indeed, we provide a theorem establishing that there exists a one-to-one correspondence between the set of feasible solutions of a ternary IP and the set of all ASWFs. Then, we exploit these generalized integer programs as a basic tool to show our characterization theorem on ASWFs with ties.

This new characterization result raises the question of which is the relationship between decomposable and strictly decomposable domains. We point out a redundant condition in the notion of decomposability proposed by Kalai and Muller (1977) and conclude our analysis showing that all strictly decomposable domains are decomposable whereas the converse relation does not hold.

2 Notation and definitions

Let E be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of E, denoted by n. Elements of E are called agents.

Let $\mathcal{E}$ be the collection of all subsets of E. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

[2] We have to stress that we still apply the basic tools of integer linear programming and that the programs we introduce could be equivalently defined on the set $\{0, 1, 2\}$. Nonetheless, here we prefer to follow Sethuraman et al. (2006), and keep using the value $\frac{1}{2}$ in order to incorporate indifference between social alternatives into the analysis.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

Let $\mathcal{R}$ be the set of all the complete and transitive binary relations on $\mathcal{A}$, called preference orderings.

Let Σ be the set of all antisymmetric preference orderings.

Let Ω denote a nonempty subset of Σ. An element of Ω is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if x is ranked above y under $\mathbf{p}$.

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let TR denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let NTR denote the set of nontrivial pairs.

Let Ω^n denote the n-fold Cartesian product of Ω. An element of Ω^n is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on Ω is a function $f : \Omega^n \to \mathcal{R}$.

f is said to be "without ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) = \emptyset$.

f is said to be "with ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) \neq \emptyset$.

Given $\mathbf{P} \in \Omega^n$, let $P(f(\mathbf{P}))$ and $I(f(\mathbf{P}))$ be binary relations on $\mathcal{A}$. We write $xP(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ but not $yf(\mathbf{P})x$ and $xI(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ and $yf(\mathbf{P})x$.

A SWF on Ω, f, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_iy$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$.

A SWF on Ω, f, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P'} \in \Omega^n$, $x\mathbf{p}_iy$ if and only if $x\mathbf{p}'_iy$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P'})y$, and $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P'})x$.

An Arrovian Social Welfare Function (ASWF) on Ω is a SWF on Ω, f, which satisfies PO and IIA.

An ASWF on Ω, f, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_jy$ implies $xP(f(\mathbf{P}))y$. f is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, \frac{1}{2}, 1\}$.

An Integer Program (IP) on Ω consists of a set of linear constraints, related to the preference orderings in Ω, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let d denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on Ω. d is said to be a binary solution if variables $d_S(x, y)$ reduce to assume values in the set $\{0, 1\}$, for all $(x, y) \in NTR$, and for all $S \in \mathcal{E}$. It is said to be a "ternary" solution, otherwise.

A solution d is dictatorial if there exists $j \in E$ such that $d_S(x, y) = 1$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$. d is nondictatorial if it is not dictatorial.

An ASWF on Ω, f, and a solution to an IP on the same Ω, d, are said to correspond if, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

3 Arrovian social welfare functions and ternary integer programming: a correspondence theorem

The first formulation of an IP on Ω was proposed by Sethuraman et al. (2003), for the case where $d_S(x, y) \in \{0, 1\}$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$. Moreover, in both their 2003 and 2006 papers, they used binary IPs on Ω to provide a representation of ASWFs different form the axiomatic one previously used in the Arrow's tradition.

In this section, we extend Sethuraman et al.'s approach, specifying two integer programs in which variables $d_S(x, y)$ are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$. We will show that these ternary programs on Ω can be used to provide a general representation of ASWFs, with and without ties in the range. Our first IP on Ω - called IP1 - consists of the following set of constraints:

$$d_E(x, y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

if $d_{A \cup U \cup V}(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup V \cup W}(z, x) \in \{0, 1\}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) = \frac{3}{2}, \tag{4}$$

if $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ or $d_{B\cup U\cup W}(y,z) = \frac{1}{2}$ or $d_{C\cup V\cup W}(z,x) = \frac{1}{2}$, for all triples of alternatives x, y, z and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy the following conditions, drawn from Sethuraman et al. (2003), and hereafter referred to as Conditions $(*)$:

$$
\begin{aligned}
A \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}z\mathbf{p}y,\\
B \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}x\mathbf{p}z,\\
C \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}y\mathbf{p}x,\\
U \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}y\mathbf{p}z,\\
V \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}x\mathbf{p}y,\\
W \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}z\mathbf{p}x.
\end{aligned}
$$

In fact, we propose now a result which establishes a one-to-one correspondence between the set of the solutions to IP1 on a given Ω and the set of all ASWFs on the same Ω.

Theorem 1. *Consider a domain Ω. Given an ASWF on Ω, f, there exists a unique solution to IP1 on Ω, d, which corresponds to f. Given a solution to IP1 on Ω, d, there exists a unique ASWF on Ω, f, which corresponds to d.*

Proof. Consider a domain Ω and an ASWF on Ω, f. Determine d as follows. Given $(x,y) \in NTR$ and $S \in \mathcal{E}$, consider $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Let $d_S(x,y) = 1$ if $xP(f(\mathbf{P}))y$, $d_S(x,y) = \frac{1}{2}$ if $xI(f(\mathbf{P}))y$, $d_S(x,y) = 0$ if $yP(f(\mathbf{P}))x$. Then, for each $(x,y) \in NTR$ and for each $S \in \mathcal{E}$, we have $xP(f(\mathbf{P}))y$ if and only if $d_S(x,y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x,y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x,y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$, as f satisfies IIA. d satisfies (1), as $f(\mathbf{P})$ satisfies PO, and (2), as $f(\mathbf{P})$ is a complete binary relation on $\mathcal{A}$, for all $\mathbf{P} \in \Omega^n$. Consider a triple x, y, z, and disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$. Moreover, consider $\mathbf{P} \in \Omega^n$. Then, by Conditions $(*)$, we have: $x\mathbf{p}_i y$, for all $i \in A \cup U \cup V$; $y\mathbf{p}_i x$, for all $i \in (A \cup U \cup V)^c$; $y\mathbf{p}_i z$, for all $i \in B \cup U \cup W$; $z\mathbf{p}_i y$, for all $i \in (B \cup U \cup W)^c$; $z\mathbf{p}_i x$, for all $i \in C \cup V \cup W$; $x\mathbf{p}_i z$, for all $i \in (C \cup V \cup W)^c$. Suppose that $d_{A\cup U\cup V}(x,y), d_{B\cup U\cup W}(y,z), d_{C\cup V\cup W}(z,x) \in \{0,1\}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > 2.$$

Then, we have $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Suppose that $d_{A\cup U\cup V}(x,y)=\frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y)+d_{B\cup U\cup W}(y,z)+d_{C\cup V\cup W}(z,x)<\frac{3}{2}.$$

Consider the following three cases. First, $d_{B\cup U\cup W}(y,z)=0$ and $d_{C\cup V\cup W}(z,x)=0$. Then, we have $zP(f(\mathbf{P}))yI(f(\mathbf{P}))x$ and $xP(f(\mathbf{P}))z$, a contradiction. Second, $d_{B\cup U\cup W}(y,z)=\frac{1}{2}$ and $d_{C\cup V\cup W}(z,x)=0$. Then, we have $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xP(f(\mathbf{P}))z$, a contradiction. Third, $d_{B\cup U\cup W}(y,z)=0$ and $d_{C\cup V\cup W}(z,x)=\frac{1}{2}$. Then, we have $zI(f(\mathbf{P}))xI(f(\mathbf{P}))y$ and $zP(f(\mathbf{P}))y$, a contradiction. Suppose now that $d_{A\cup U\cup V}(x,y)=\frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y)+d_{B\cup U\cup W}(y,z)+d_{C\cup V\cup W}(z,x)>\frac{3}{2}.$$

Consider the following three cases. First, $d_{B\cup U\cup W}(y,z)=1$ and $d_{C\cup V\cup W}(z,x)=1$. Then, we have $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Second, $d_{B\cup U\cup W}(y,z)=\frac{1}{2}$ and $d_{C\cup V\cup W}(z,x)=1$. Then, we have $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Third, $d_{B\cup U\cup W}(y,z)=1$ and $d_{C\cup V\cup W}(z,x)=\frac{1}{2}$. Then, we have $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zI(f(\mathbf{P}))x$, a contradiction. Therefore, d satisfies (3) and (4). Hence, d is a solution to IP1 on Ω which corresponds to f. Suppose that d is not unique. Then, there exist a solution to IP1 on Ω, d', $(x,y)\in NTR$, and $S\in\mathcal{E}$ such that $d_S(x,y)\neq d'_S(x,y)$. Consider $\mathbf{P}\in\Omega^n$ such that $x\mathbf{p}_iy$, for all $i\in S$, and $y\mathbf{p}_ix$, for all $i\in S^c$. Then, we have $xP(f(\mathbf{P}))y$ and $xI(f(\mathbf{P}))y$, or, $yP(f(\mathbf{P}))x$ and $xI(f(\mathbf{P}))y$, or, $xP(f(\mathbf{P}))y$ and $yP(f(\mathbf{P}))x$, a contradiction. But then, d is unique. Now, consider a solution to IP1 on Ω, d. Determine f as follows. Given $(x,y)\in TR$, let $xP(f(\mathbf{P}))y$, for all $\mathbf{P}\in\Omega^n$. Given $(x,y)\in NTR$ and $\mathbf{P}\in\Omega^n$, let $S\in\mathcal{E}$ be the set of agents such that $x\mathbf{p}_iy$, for all $i\in S$, and $y\mathbf{p}_ix$, for all $i\in S^c$. Let $xP(f(\mathbf{P}))y$ if $d_S(x,y)=1$, $xI(f(\mathbf{P}))y$ if $d_S(x,y)=\frac{1}{2}$, and $yP(f(\mathbf{P}))x$ if $d_S(x,y)=0$. $f(\mathbf{P})$ is a complete binary relation on $\mathcal{A}$, for all $\mathbf{P}\in\Omega^n$, by construction and by (2). Now, we show that $f(\mathbf{P})$ is also a transitive binary relation on $\mathcal{A}$, for all $\mathbf{P}\in\Omega^n$. Consider a triple x,y,z and a preference profile $\mathbf{P}\in\Omega^n$. Then, there exist three nonempty sets H, I, J such that $x\mathbf{p}_iy$, for all $i\in H$, $y\mathbf{p}_ix$, for all $i\in H^c$, $y\mathbf{p}_iz$, for all $i\in I$, $z\mathbf{p}_iy$, for all $i\in I^c$, $z\mathbf{p}_ix$, for all $i\in J$, $x\mathbf{p}_iz$, for all $i\in J^c$. Let $A=H\setminus(I\cup J)$, $B=I\setminus(H\cup J)$, $C=J\setminus(H\cup I)$, $U=H\cap I$, $V=H\cap J$, $W=I\cap J$. Then, $A,B,C,U,V,W\in\mathcal{E}$ are disjoint sets of agents whose union includes all agents and which satisfy Conditions $(*)$. Moreover, they satisfy $A \cup U \cup V = H$, $B \cup U \cup W = I$, $C \cup V \cup W = J$. Consider the following eight cases. First, $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A \cup U \cup V}(x,y) = 1$, $d_{B \cup U \cup W}(y,z) = 1$, $d_{C \cup V \cup W}(z,x) = 1$, and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > 2,$$

contradicting (3). Second, $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{C \cup V \cup W}(z,x) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Third, $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A \cup U \cup V}(x,y) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Fourth, $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{A \cup U \cup V}(x,y) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Fifth, $xP(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{B \cup U \cup W}(y,z) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Sixth, $xP(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{B \cup U \cup W}(y,z) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Seventh, $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xP(f(\mathbf{P}))z$. Then, $d_{A \cup U \cup V}(x,y) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) < \frac{3}{2},$$

contradicting (4). Eighth, $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A \cup U \cup V}(x,y) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) > \frac{3}{2},$$

contradicting (4). f satisfies PO as, for all $(x, y) \in TR$, we have $xP(f(\mathbf{P}))y$, for all $\mathbf{P} \in \Omega^n$; moreover, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_i y$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$, by (1). f satisfies IIA as, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, we have $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, and $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Hence, f is an ASWF on Ω, which corresponds to d. Suppose that f is not unique. Then, there exists an ASWF on Ω, f', $(x, y) \in NTR$ and $\mathbf{P} \in \Omega^n$ such that we have $xf(\mathbf{P})y$ but not $xf'(\mathbf{P})y$. Let $S \in \mathcal{E}$ be the set such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Then, $d_S(x, y) = 1$ and $d_S(x, y) = 0$, or, $d_S(x, y) = \frac{1}{2}$ and $d_S(x, y) = 0$, a contradiction. But then, f is unique. ∎

We introduce now a second ternary IP on Ω, which we will call IP2. It consists of constraints (1), (2), and the following four logically independent constraints:[3]

$$d_S(x, y) \leq d_S(x, z), \tag{5}$$

if $d_S(x, y) \in \{0, 1\}$;

$$d_S(x, y) < d_S(x, z), \tag{6}$$

if $d_S(x, y) = \frac{1}{2}$, for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x, y) + d_S(y, z) \leq 1 + d_S(x, z), \tag{7}$$

if $d_S(x, y), d_S(y, z) \in \{0, 1\}$;

$$d_S(x, y) + d_S(y, z) = \frac{1}{2} + d_S(x, z), \tag{8}$$

if $d_S(x, y) = \frac{1}{2}$ or $d_S(y, z) = \frac{1}{2}$, for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, and for all $S \in \mathcal{E}$.

In the remainder of this section, we prove two propositions which establish the relationships between IP1 and IP2.

[3]In building IP2, we take inspiration from a binary IP on Ω, introduced by Sethuraman et al. (2003), which incorporates a reformulation of Kalai and Muller's condition of decomposability. It can be shown that the set of constraints proposed by Sethuraman et al. exhibits problems of logical dependence (see Busetto and Codognato (2010)), which are eliminated in our IP2. These problems parallel some logical redundancies inherent in Kalai and Muller's notion of decomposability, which we will point out in Section 4.

Proposition 1. *If d is a solution to IP1 on Ω, then it is a solution to IP2 on the same Ω.*

Proof. Let d be a solution to IP1 on Ω. Consider a triple x, y, z and $S \in \mathcal{E}$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ which satisfy $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Let $U = S$, $W = S^c$, and $A = B = C = V = \emptyset$. Then, A, B, C, U, V, W are sets whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_S(x, y) \in \{0, 1\}$ and $d_S(x, y) > d_S(x, z)$. Consider the following two cases. First, $d_S(x, z) \in \{0, 1\}$. Then,

$$d_U(x, y) + d_{U\cup W}(y, z) + d_W(z, x) > 2,$$

contradicting (3). Second, $d_S(x, z) = \frac{1}{2}$. Then,

$$d_U(x, y) + d_{U\cup W}(y, z) + d_W(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, d satisfies (5). Suppose now that $d_S(x, y) = \frac{1}{2}$ and $d_S(x, y) \geq d_S(x, z)$. Then,

$$d_U(x, y) + d_{U\cup W}(y, z) + d_W(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, d satisfies (6). Consider a triple x, y, z and $S \in \mathcal{E}$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Let $C = S^c$, $U = S$, and $A = B = V = W = \emptyset$. Then, A, B, C, U, V, W are sets whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_S(x, y), d_S(y, z) \in \{0, 1\}$ and $d_S(x, y) + d_S(y, z) > 1 + d_S(x, z)$. Consider the following two cases. First, $d_S(x, z) \in \{0, 1\}$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) > 2,$$

contradicting (3). Second, $d_S(x, z) = \frac{1}{2}$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, d satisfies (7). Suppose now that $d_S(x, y) = \frac{1}{2}$ and $d_S(x, y) + d_S(y, z) < \frac{1}{2} + d_S(x, z)$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) < \frac{3}{2},$$

contradicting (4). Suppose that $d_S(x,y) = \frac{1}{2}$ and $d_S(x,y) + d_S(y,z) > \frac{1}{2} + d_S(x,z)$. Then,

$$d_U(x,y) + d_U(y,z) + d_C(z,x) > \frac{3}{2},$$

contradicting (4). Therefore, d satisfies (8). Hence, d is a solution to IP2 on Ω. ■

The following result shows that the converse of Proposition 3 holds - and IP1 and IP2 coincide - when $n = 2$.

Proposition 2. *Let $n = 2$. If d is a solution to IP2 on Ω, then it is a solution to IP1 on the same Ω.*

Proof. Let $n = 2$. Let d be a solution to IP2 on Ω. Consider a triple x, y, z and disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_{A\cup U\cup V}(x,y), d_{B\cup U\cup W}(y,z), d_{C\cup V\cup W}(z,x) \in \{0,1\}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > 2.$$

Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Then,

$$d_{\{2\}}(y,z) + d_{\{2\}}(z,x) > 1 + d_{\{2\}}(y,x),$$

contradicting (7). The cases where $B \neq \emptyset$, $V \neq \emptyset$, and $C \neq \emptyset$, $U \neq \emptyset$ lead, *mutatis mutandis*, to the same contradiction. Consider the case where $U \neq \emptyset$ and $V \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}x\mathbf{q}y$. Suppose that $U = \{1\}$ and $V = \{2\}$. Then,

$$d_{\{2\}}(z,x) > d_{\{2\}}(z,y),$$

contradicting (5). The cases where $V \neq \emptyset$, $W \neq \emptyset$, and $U \neq \emptyset$, $W \neq \emptyset$, lead, *mutatis mutandis*, to the same contradiction. Therefore, d satisfies (3). Suppose that $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) < \frac{3}{2}.$$

Consider the case where $A \neq \emptyset$ and $B \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}x\mathbf{q}z$. Suppose that $A = \{1\}$ and $B = \{2\}$. Then, $d_{\{2\}}(y,x) = \frac{1}{2}$ and

$$d_{\{2\}}(y,x) \geq d_{\{2\}}(y,z),$$

contradicting (6). The case where $A \neq \emptyset$ and $C \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Suppose that $d_{\{2\}}(y, z) = 0$ and $d_{\{2\}}(z, x) = 0$. Then,

$$d_{\{1\}}(x, z) + d_{\{1\}}(z, y) > 1 + d_{\{1\}}(x, y),$$

contradicting (7). Suppose that $d_{\{2\}}(y, z) = \frac{1}{2}$ and $d_{\{2\}}(z, x) = 0$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) < \frac{1}{2} + d_{\{2\}}(y, x),$$

contradicting (8). Consider the case where $U \neq \emptyset$ and $C \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Suppose that $U = \{1\}$ and $C = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) < \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). The case where $V \neq \emptyset$ and $B \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Suppose that $d_{A \cup U \cup V}(x, y) = \frac{1}{2}$ and

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) > \frac{3}{2}.$$

Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Suppose that $d_{\{2\}}(y, z) = 1$ and $d_{\{2\}}(z, x) = 1$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) > 1 + d_{\{2\}}(y, x),$$

contradicting (7). Suppose that $d_{\{2\}}(y, z) = \frac{1}{2}$ and $d_{\{2\}}(z, x) = 1$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) > \frac{1}{2} + d_{\{2\}}(y, x),$$

contradicting (8). Consider the case where $U \neq \emptyset$ and $C \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Suppose that $U = \{1\}$ and $C = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). The case where $V \neq \emptyset$ and $B \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Consider the case where $U \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $U = \{1\}$ and $W = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z),$$

contradicting (6). The case where $V \neq \emptyset$ and $W \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Therefore, d satisfies (4). Hence, d is a solution to IP1 on Ω. ■

4 Nondictatorial Arrovian social welfare functions with ties and integer programming: a new characterization theorem

In this section, we use the integer programs developed above to deal with the issues concerning the dictatorship property of ASWFs. As already reminded, Arrow's impossibility theorem is established for ASWFs admitting ties in their range and defined on the unrestricted domain of preference orderings.

Kalai and Muller (1977) were the first who overcome Arrow's impossibility theorem by providing a complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs without ties. They did this by means of two theorems. In their Theorem 1, they showed that, for a given domain Ω, there exists a nondictatorial ASWF without ties for $n > 2$ if and only if, for the same Ω, there exists a nondictatorial ASWF without ties for $n = 2$. In their Theorem 2, they gave the domain characterization, based on the following notion of decomposability, henceforth called KM-decomposability.

Ω is said to be KM-decomposable if there exists a set R, with $TR \subsetneq R \subsetneq \mathcal{A}^2$, satisfying the following conditions.

Condition I. For every two pairs $(x, y), (x, z) \in NTR$*, if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$*, then* $(x, y) \in R$ *implies that* $(x, z) \in R$*.*

Condition II. For every two pairs $(x, y), (x, z) \in NTR$*, if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$*, then* $(z, x) \in R$ *implies that* $(y, x) \in R$*.*

Condition III. For every two pairs $(x, y), (x, z) \in NTR$*, if there exists* $\mathbf{p} \in \Omega$ *for which* $x\mathbf{p}y\mathbf{p}z$*, then* $(x, y) \in R$ *and* $(y, z) \in R$ *imply that* $(x, z) \in R$*.*

Condition IV. For every two pairs $(x, y), (x, z) \in NTR$, if there exists $\mathbf{p} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$, then $(z, x) \in R$ implies that $(y, x) \in R$ or $(z, y) \in R$.

It is useful to reproduce here Kalai and Muller's characterization theorem for ASWFs without ties. It can be stated as follows.

Theorem 2. *There exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$, if and only if Ω is KM-decomposable.*

The fundamental aim of this section is taking a step forward along the way opened by Kalai and Muller: our main theorem establishes a characterization of the domains of antisymmetric preference orderings admitting nondictatorial ASWFs with ties.

In order to prove it, we need to establish some preliminary results. To begin with, let us reconsider Kalai and Muller's Theorem 1: Sethuraman et al. (2003) provided a reformulation of this theorem in terms of integer programming. More precisely, they established a be-univocal relation between the nondictatorial solutions of a binary IP on Ω, for $n = 2$, and its nondictatorial solutions for $n > 2$. Here, we extend this result to the case of ternary solutions to IP1.

Theorem 3. *There exists a nondictatorial ternary solution to IP1 on Ω, d, for $n = 2$, if and only if there exists a nondictatorial ternary solution to IP1 on Ω, d^*, for $n > 2$.*

Proof. Let d be a nondictatorial ternary solution to IP1 on Ω for $n = 2$. Determine d^* as follows. Given $(x, y) \in NTR$ and $S \in \mathcal{E}$, let $d^*_S(x, y) = 1$ if $1, 2 \in S$; $d_S(x, y) = 0$ if $1, 2 \in S^c$; $d^*_S(x, y) = d_{\{1\}}(x, y)$ and $d^*_{S^c}(y, x) = d_{\{2\}}(y, x)$ if $1 \in S$ and $2 \in S^c$. Then, it is straightforward to verify that d^* satisfies (1)-(4) and that is nondictatorial. Hence, d^* is a nondictatorial ternary solution to IP1 on Ω, for $n > 2$. Conversely, let d^* be a nondictatorial ternary solution to IP1 on Ω for $n > 2$. Determine d as follows. Consider $(u, v) \in NTR$ and $\bar{S} \in \mathcal{E}$ such that $d^*_{\bar{S}}(u, v) = \frac{1}{2}$. Given $(x, y) \in NTR$, let $d_{\{1,2\}}(x, y) = 1$, $d_\emptyset(x, y) = 0$, $d_{\{1\}}(x, y) = d^*_{\bar{S}}(x, y)$, $d_{\{2\}}(y, x) = d^*_{\bar{S}^c}(y, x)$. Then, it is straightforward to verify that d satisfies (1) and (2). Moreover, by Proposition 1, d satisfies (5)-(8) as d^* is a solution to IP1 on Ω. But then, d is a solution to IP2 on Ω and this, in turn, implies that it is a solution to IP1 on Ω, by Proposition 2. Finally, d is nondictatorial as $d_{\{1\}}(u, v) = \frac{1}{2}$. Hence, d is a nondictatorial ternary solution to IP1 on Ω, for $n = 2$. ■

From Theorem 3, we obtain the following corollary, which extends Kalai and Muller's Theorem 1 to the case of ASWFs with ties. It is an immediate consequence of our Theorem 1 in Section 3.

Corollary. *There exists a nondictatorial ASWF with ties on Ω, f, for $n = 2$, if and only if there exists a nondictatorial ASWF with ties on Ω, f^*, for $n > 2$.*

At this point, we need to introduce a reformulation of the concept of KM-decomposability suitable to be applied within the analytical context of a ternary IP on Ω. We will show below that this reformulation is equivalent to the original version proposed by Kalai and Muller. Our concept is based on the existence of two sets, $R_1, R_2 \in \mathcal{A}^2$ - instead of only one - satisfying the restrictions introduced here.

Given a set $R \subset \mathcal{A}^2$, consider the following conditions on R.

Condition 1. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition 2. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be decomposable if there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2.

With regard to this definition of a decomposable domain, let us notice the main differences with Kalai and Muller's original notion, introduced to make it compatible with the integer programming analytical setting: Conditions 1 and 2 differ from the corresponding Conditions I and III as the former refer to triples, rather than pairs, of alternatives. Moreover, Condition 2 is reformulated in terms of a pair of preference orderings, instead of only one. This is consistent with the formulation of our constraints (7) and (8), which are in fact a reinterpretation of Condition 2 in terms of integer programming. Also, our notion of decomposability does not require that R_1 and R_2 contain TR, whereas Kalai and Muller's one requires that R contains TR. In particular, let us stress that our definition requires that R_1 and R_2 satisfy only two conditions - instead of four, as in Kalai and Muller's version. As the next proposition makes it clear, this implies a redundancy of Kalai and Muller's Conditions II and IV. Nevertheless, as anticipated above, the following proposition establishes that the two concepts are equivalent.

Proposition 3. *Ω is KM-decomposable if and only if it is decomposable.*

Proof. Let Ω be KM-decomposable. Then, there exists a set R, with $TR \subsetneqq R \subsetneqq \mathcal{A}^2$, which satisfies Conditions I-IV. By Lemma 4 in Kalai and Muller, there exists a set $\bar{R}$, with $TR \subsetneq \bar{R} \subsetneq \mathcal{A}^2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R$ if and only if $(y, x) \notin \bar{R}$, and which satisfies Conditions I-IV. Let $R_1 = R \setminus TR$ and $R_2 = \bar{R} \setminus TR$. Then, $\emptyset \subsetneq R_i \subsetneq NTR$, $i = 1, 2$, and, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$. Consider a triple x, y, z and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x, y) \in R_1$ and $(x, z) \notin R_1$. Then, $(x, y) \in R$ and $(x, z) \notin R$ as $(x, z) \in NTR$, contradicting Condition I. Hence, R_i, $i = 1, 2$, satisfies Condition 1. Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x, y), (y, z) \in R_1$ and $(x, z) \notin R_1$. Then, $(x, y), (y, z) \in R$, and $(x, z) \notin R$ as $(x, z) \in NTR$, contradicting Condition III. Hence, R_i, $i = 1, 2$, satisfies Condition 2. We have proved that Ω is decomposable. Conversely, suppose that Ω is decomposable. Then, there exist two sets R_1 and R_2, with $\emptyset \subsetneq R_i \subsetneq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2. Let $R = R_1 \cup TR$. Consider two pairs $(x, y), (x, z) \in NTR$ and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x, y) \in R$ and $(x, z) \notin R$. Then, $(x, y) \in R_1$ and $(x, z) \notin R_1$ as $(x, y), (x, z) \in NTR$, contradicting Condition 1. Hence, R satisfies Condition I. Now, suppose that $(z, x) \in R$ and $(y, x) \notin R$. Then, $(x, y) \in R_2$ and $(x, z) \notin R_2$ as $(x, y), (x, z) \in NTR$, contradicting Condition 1. Hence, R satisfies Condition II. Consider two pairs $(x, y), (x, z) \in NTR$ and suppose there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$. Moreover, suppose that $(x, y), (y, z) \in R$, and $(x, z) \notin R$. There exists $\mathbf{q} \in \Omega$ such that $z\mathbf{q}x$ as $(x, z) \in NTR$. Consider the case where $y\mathbf{q}z\mathbf{q}x$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, $(x, y) \in R$, and $(x, z) \notin R$, contradicting Condition I. Consider the case where $z\mathbf{q}x\mathbf{q}y$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}x\mathbf{q}y$, $(y, z) \in R$, and $(x, z) \notin R$, contradicting Condition II. Consider the case where $z\mathbf{q}y\mathbf{q}x$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, $(x, y), (y, z) \in R_1$, and $(x, z) \notin R_1$ as $(x, y), (y, z), (x, z) \in NTR$, contradicting Condition 2. Hence, R satisfies Condition III. Consider two pairs $(x, y), (x, z) \in NTR$ and suppose there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$. Moreover, suppose that $(z, x) \in R$ and $(y, x), (z, y) \notin R$. There exists $\mathbf{q} \in \Omega$ such that $z\mathbf{q}x$ as $(x, z) \in NTR$. Consider the case where $z\mathbf{q}x\mathbf{q}y$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}x\mathbf{q}y$, $(z, x) \in R$, and $(z, y) \notin R$, contradicting Condition I. Consider the case where $y\mathbf{q}z\mathbf{q}x$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, $(z, x) \in R$, and $(y, x) \notin R$, contradicting Condition II. Consider the case where $z\mathbf{q}x\mathbf{q}y$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, $(x, y), (y, z) \in R_2$, and $(x, z) \notin$

R_2 as $(x, y), (y, z), (x, z) \in NTR$, contradicting Condition 2. Hence, R satisfies Condition IV. We have proved that Ω is KM-decomposable. ■

In order to obtain our characterization theorem for ASWFs with ties, we need to restrict further the condition of decomposability introduced above. Then, we introduce a new notion, which we define as "strict decomposability." The next section will be devoted to establish the exact relationship between the two notions of decomposability and strict decomposability.

Then, given a set $R \subset \mathcal{A}^2$, consider the following conditions on R.

Condition 3. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R^*$ implies that $(x, z) \in R$.

Condition 4. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples of alternatives x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R^*$ imply that $(x, z) \in R$, and $(x, y) \in R^*$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be strictly decomposable if and only if there exist four sets R_1, R_2, R_1^*, and R_2^*, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, R_i, $i = 1, 2$, satisfies Condition 1; R_i and R_i^*, $i = 1, 2$, satisfy Condition 2; each pair (R_i, R_i^*), $i = 1, 2$, satisfies Conditions 3 and 4.

On the basis of the notion of strict decomposability, we provide now the characterization of domains admitting nondictatorial ternary solutions to IP1.

Theorem 4. *There exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$, if and only if Ω is strictly decomposable.*

Proof. Let d be a nondictatorial ternary solution to IP2 on Ω, for $n = 2$. Let $R_1 = \{(x, y) \in NTR : d_{\{1\}}(x, y) = 1\}$, $R_2 = \{(x, y) \in NTR : d_{\{2\}}(x, y) = 1\}$, $R_1^* = \{(x, y) \in NTR : d_{\{1\}}(x, y) = \frac{1}{2}\}$, $R_2^* = \{(x, y) \in NTR : d_{\{2\}}(x, y) = \frac{1}{2}\}$. Consider $(x, y) \in NTR$. Suppose that $(x, y) \in R_1$ and $(x, y) \in R_1^*$. Then, $d_{\{1\}}(x, y) = 1$ and $d_{\{1\}}(x, y) = \frac{1}{2}$, a contradiction. Suppose that $(x, y) \in R_1$ and $(y, x) \in R_2$. Then, $d_{\{1\}}(x, y) = 1$ and $d_{\{2\}}(y, x) = 1$, contradicting (2). Suppose that $(x, y) \notin R_1^*$, $(y, x) \notin R_2$, and $(x, y) \notin R_1$. Then, $d_{\{1\}}(x, y) \neq \frac{1}{2}$, $d_{\{1\}}(x, y) \neq 0$, and $d_{\{1\}}(x, y) \neq 1$, a contradiction. Suppose that $(x, y) \in R_1^*$ and $(y, x) \notin R_2^*$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$

and $d_{\{2\}}(y,x) \neq \frac{1}{2}$, contradicting (2). Hence, for all $(x,y) \in NTR$, $(x,y) \in R_1$ if and only if $(x,y) \notin R_1^*$ and $(y,x) \notin R_2$; $(x,y) \in R_1^*$ if and only if $(y,x) \in R_2^*$. Suppose that $R_1 = NTR$. Then, d is dictatorial, a contradiction. Hence, $R_i \subsetneqq NTR$, $i = 1,2$. Suppose that $R_i^* = \emptyset$, $i = 1,2$. Then, d is a binary solution, a contradiction. Hence, $\emptyset \subsetneqq R_i^* \subset NTR$. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1$ and $(x,z) \notin R_1$ Then, $d_{\{1\}}(x,y) = 1$ and

$$d_{\{1\}}(x,y) > d_{\{1\}}(x,z),$$

contradicting (5). Hence, R_i, $i = 1,2$, satisfies Condition 1. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x,y),(y,z) \in R_1$, and $(x,z) \notin R_1$. Then, $d_{\{1\}}(x,y) = 1$, $d_{\{1\}}(y,z) = 1$, and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) > 1 + d_{\{1\}}(x,z),$$

contradicting (7). Hence, R_i, $i = 1,2$, satisfies Condition 2. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1^*$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1^*$. Then, $d_{\{1\}}(x,y) = \frac{1}{2}$, $d_{\{1\}}(y,z) = \frac{1}{2}$, and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) \neq \frac{1}{2} + d_{\{1\}}(x,z),$$

contradicting (8). Hence, R_i^* satisfies Condition 2, $i = 1,2$. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1^*$ and $(x,z) \notin R_1$. Then, $d_{\{1\}}(x,y) = \frac{1}{2}$ and

$$d_{\{1\}}(x,y) \geq d_{\{1\}}(x,z),$$

contradicting (6). Hence, each pair (R_i, R_i^*), $i = 1,2$, satisfies Condition 3. Consider a triple x,y,z and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1$. Then, $d_{\{1\}}(y,z) = \frac{1}{2}$ and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) \neq \frac{1}{2} + d_{\{1\}}(x,z),$$

contradicting (8). Now, suppose that $(x,y) \in R_1^*$, $(y,z) \in R_1$, and $(x,z) \notin R_1$. Then, $d_{\{1\}}(x,y) = \frac{1}{2}$ and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) \neq \frac{1}{2} + d_{\{1\}}(x,z),$$

contradicting (8). Hence, each pair (R_i, R_i^*), $i = 1, 2$, satisfies Condition 4. We have proved that Ω is strictly decomposable. Conversely, suppose that Ω is strictly decomposable. Then, there exist four sets R_1, R_2, R_1^*, and R_2^*, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, R_i, $i = 1, 2$, satisfies Condition 1; R_i and R_i^*, $i = 1, 2$, satisfy Condition 2; each pair (R_i, R_i^*), $i = 1, 2$, satisfies Conditions 3 and 4. Determine d as follows. For each $(x, y) \in NTR$, let $d_\emptyset(x, y) = 0$, $d_E(x, y) = 1$; $d_{\{i\}}(x, y) = 1$ if and only if $(x, y) \in R_i$; $d_{\{i\}}(x, y) = \frac{1}{2}$ if and only if $(x, y) \in R_i^*$; $d_{\{i\}}(x, y) = 0$ if and only if, $(x, y) \notin R_i$ and $(x, y) \notin R_i^*$, for $i = 1, 2$. Then, d satisfies (1) and (2) as, for all $(x, y) \in NTR$, $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$, $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$. Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z).$$

Then, we have $(x, y) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 1. Therefore, d satisfies (5). Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z).$$

Then, we have $(x, y), (y, z) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 2. Therefore, d satisfies (7). Consider a triple x, y, z and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z).$$

Then, $(x, y) \in R_1^*$ and $(x, z) \notin R_1$, contradicting Condition 3. Therefore, d satisfies (6). Consider a triple x, y, z and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z).$$

Consider the following two cases. First, $d_{\{1\}}(y, z) = 1$. Then, $(x, y) \in R_1^*$, $(y, z) \in R_1$, and $(x, z) \notin R_1$, contradicting Condition 4. Second, $d_{\{1\}}(y, z) = \frac{1}{2}$. Then, $(x, y) \in R_1^*$, $(y, z) \in R_1^*$, and $(x, z) \notin R_1^*$, contradicting Condition 2. Finally, suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) < \frac{1}{2} + d_{\{1\}}(x, z).$$

Consider the following two cases. First, $d_{\{1\}}(y,z) = 0$. Then, $(z,y) \in R_2$, $(y,x) \in R_2^*$, and $(z,x) \notin R_2$, contradicting Condition 4. Second, $d_{\{1\}}(y,z) = \frac{1}{2}$. Then, $(x,y) \in R_1^*$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1^*$, contradicting Condition 2. Therefore, d satisfies (8). d is nondictatorial as $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1,2$. Hence, d is a nondictatorial ternary solution to IP2 on Ω. ■

Our characterization theorem for ASWFs with ties immediately follows from Theorems 1 and 3. This result is a generalization of Kalai and Muller's Theorem 2 for ASWFs without ties.

Theorem 5. *There exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$, if and only if Ω is strictly decomposable.*

Proof. It is a straightforward consequence of Propositions 1 and 2, Theorems 1, 3, and 4. ■

5 The relationship between decomposable and strictly decomposable domains

In this section, we analyze the relationship between the notions of decomposable and strictly decomposable domain. The following example illustrates the two notions.

Example 1. *Let $A = \{a,b,c,d\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c\mathbf{p}d, c\mathbf{p}d\mathbf{p}a\mathbf{p}b, d\mathbf{p}c\mathbf{p}b\mathbf{p}a\}$. Then, Ω is decomposable and strictly decomposable.*

Proof. The triples x,y,z for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$ are c,a,b; d,a,b; a,c,d; b,c,d. The triples x,y,z for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$ are a,b,c; a,b,d; a,c,d; b,c,d. Let $R_1 = \{(a,b),(b,a),(c,d),(d,c)\}$ and $R_2 = \{(a,c),(c,a),(a,d),(d,a),(b,c),(c,b),(b,d),(d,b)\}$. Then, we have $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. R_1 vacuously satisfies Conditions 1 and 2. R_2 satisfies Condition 1 as we have: $(a,c) \in R_2$ and $(a,d) \in R_2$; $(c,a) \in R_2$ and $(c,b) \in R_2$; $(d,a) \in R_2$ and $(d,b) \in R_2$; $(b,c) \in R_2$ and $(b,d) \in R_2$. R_2 vacuously satisfies Condition 2. We have shown that Ω is decomposable. Now, let $V_1 = \{(a,b),(c,d)\}$, $V_2 = \{(a,c),(c,a),(a,d),(d,a),(b,c),(c,b),(b,d),(d,b)\}$, $V_1^* = \{(b,a),(d,c)\}$, $V_2^* = \{(a,b),(c,d)\}$. Then, we have $V_i \subsetneqq NTR$, $i = 1,2$, and $\emptyset \subsetneqq V_i^* \subset NTR$, $i = 1,2$. Moreover, for all $(x,y) \in NTR$, we have: $(x,y) \in V_1$ if and only if $(x,y) \notin V_1^*$ and $(y,x) \notin V_2$; $(x,y) \in V_1^*$ if and only if $(y,x) \in V_2^*$. V_1

vacuously satisfies Conditions 1 and 2. V_1^* vacuously satisfies Condition 2. Moreover, the pair (V_1, V_1^*) vacuously satisfies Conditions 3 and 4. V_2 satisfies Conditions 1 and 2 as $V_2 = R_2$. V_2^* vacuously satisfies Condition 2. The pair (V_2, V_2^*) vacuously satisfies Condition 3. Moreover, it satisfies Condition 4 as we have: $(a, c) \in V_2$, $(c, d) \in V_2^*$, and $(a, d) \in V_2$; $(b, c) \in V_2$, $(c, d) \in V_2^*$, and $(b, d) \in V_2$; $(a, b) \in V_2^*$, $(b, c) \in V_2$, and $(a, c) \in V_2$; $(a, b) \in V_2^*$, $(b, d) \in V_2$, and $(a, d) \in V_2$. We have shown that Ω is strictly decomposable. ■

The example above specifies a domain which is both decomposable and strictly decomposable. Nonetheless, this is not the general case. In the following, we will show, with a theorem and a further example, that a strictly decomposable domain is always decomposable, but the converse is not true.

In order to obtain these results, we preliminarily show the following theorem on the nondictatorial solutions to IP2.

Theorem 6. *If there exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$, then there exists a nondictatorial binary solution to IP2 on Ω, $\hat{d}$, for $n = 2$.*

Proof. Let d be a ternary solution to IP2 on Ω, for $n = 2$. Determine d' as follows. Consider $\mathbf{q} \in \Sigma$. For each $(x, y) \in NTR$, let: $d'_{\emptyset}(x, y) = 0$, $d'_E(x, y)$=1; $d'_{\{i\}}(x, y) = d_{\{i\}}(x, y)$, if $d_{\{i\}}(x, y) \in \{0, 1\}$, $i = 1, 2$; $d'_{\{1\}}(x, y) = 1$ and $d'_{\{2\}}(y, x) = 0$, if $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$ and $x\mathbf{q}y$. Then, it is immediate to verify that d' is a solution to IP2 on Ω, for $n = 2$. Suppose that d' is nondictatorial. Then, $\hat{d} = d'$ is a nondictatorial binary solution to IP2 on Ω, for $n = 2$. Suppose that d' is dictatorial: say, for example, that, for all $(x, y) \in NTR$, $d_S(x, y) = 1$, for all S containing agent 1. In this case, we can say that agent 1 is the dictator for d'. Determine d'' as follows. Let $\mathbf{q}^{-1} \in \Sigma$ be an antisymmetric preference ordering such that, for each $(x, y) \in \mathcal{A}^2$, $x\mathbf{q}y$ if and only if $y\mathbf{q}^{-1}x$. For each $(x, y) \in NTR$, let: $d''_{\emptyset}(x, y) = 0$, $d''_E(x, y)$=1; $d''_{\{i\}}(x, y) = d_{\{i\}}(x, y)$, if $d_{\{i\}}(x, y) \in \{0, 1\}$, $i = 1, 2$; $d''_{\{1\}}(x, y) = 1$ and $d''_{\{2\}}(y, x) = 0$, if $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$ and $x\mathbf{q}^{-1}y$. Then, it is immediate to verify that $\hat{d} = d''$ is a binary solution to IP2 on Ω, for $n = 2$, and that agent 1 is not a dictator for d''. Suppose that agent 2 is a dictator for d''. Consider $(x, y) \in NTR$ such that $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$. Suppose that $y\mathbf{q}x$. This implies that $d'_{\{1\}}(x, y) = 0$ and agent 1 is not a dictator for d', a contradiction. But then, we must have that $x\mathbf{q}y$. Consider variables $d_{\{1\}}(y, x)$ and $d_{\{2\}}(x, y)$. Suppose that $d_{\{1\}}(y, x) = 1$ and $d_{\{2\}}(x, y) = 0$. Then, agent 2 is not a dictator for d'', a contradiction.

Suppose that $d_{\{1\}}(y,x) = 0$ and $d_{\{2\}}(x,y) = 1$. Then, agent 1 is not a dictator for d'. This implies that $d_{\{1\}}(y,x) = d_{\{2\}}(x,y) = \frac{1}{2}$ and this, in turn, implies that $d''_{\{2\}}(x,y) = 0$ and agent 2 is not a dictator of d'', a contradiction. Then, $\hat{d} = d''$ is a nondictatorial binary solution to IP2 on Ω, for $n = 2$. ■

Then, the following theorem can be immediately proved.

Theorem 7. *If a domain Ω is strictly decomposable, then it is decomposable.*

Proof. Let Ω be a strictly decomposable domain. Then, by Theorem 4, there exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$. But then, by Theorem 6, there exists a nondictatorial binary solution to IP2 on Ω, $\hat{d}$, for $n = 2$. Hence, by Theorems 1 and 2, and Proposition 3, Ω is decomposable. ■

The following example shows that the converse of Theorem 7 does not hold.

Example 2. *Let $A = \{a,b,c,d\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c\mathbf{p}d, c\mathbf{p}a\mathbf{p}d\mathbf{p}b,$ $d\mathbf{p}c\mathbf{p}b\mathbf{p}a, b\mathbf{p}d\mathbf{p}a\mathbf{p}c\}$. Then, Ω is decomposable but it is not strictly decomposable.*

Proof. The triples x,y,z for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$ are: c,a,b; c,b,a; a,b,d; a,d,b,; d,a,c; d,c,a; b,c,d; b,d,c. The triples x,y,z for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$ are: a,b,c; c,a,b; a,b,d; a,d,b; a,c,d; c,a,d; b,c,d; c,d,b. Let $R_i = \{(a,b),(a,c),(a,d),$ $(b,c),(b,d),(c,d)\}$, $i = 1,2$. Then, we have $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. R_i, $i = 1,2$, satisfies Condition 1 as we have: $(a,b) \in R_i$ and $(a,d) \in R_i$; $(a,d) \in R_i$ and $(a,b) \in R_i$; $(b,c) \in R_i$ and $(b,d) \in R_i$; $(b,d) \in R_i$ and $(b,c) \in R_i$, $i = 1,2$. R_i, $i = 1,2$, satisfies Condition 2 as we have: $(a,b) \in R_i$, $(b,c) \in R_i$, and $(a,c) \in R_i$; $(a,b) \in R_i$, $(b,d) \in R_i$, and $(a,d) \in R_i$; $(a,c) \in$ R_i, $(c,d) \in R_i$, and $(a,d) \in R_i$; $(b,c) \in R_i$, $(c,d) \in R_i$, and $(b,d) \in R_i$, $i = 1,2$. We have shown that Ω is decomposable. Now suppose that Ω is strictly decomposable. Then, there exist four sets V_1, V_2, V_1^*, and V_2^*, with $V_i \subsetneqq NTR$, $\emptyset \subsetneqq V_i^* \subset NTR$, $i = 1,2$, such that, for all $(x,y) \in NTR$, we have: $(x,y) \in V_1$ if and only if $(x,y) \notin V_1^*$ and $(y,x) \notin V_2$; $(x,y) \in V_1^*$ if and only if $(y,x) \in V_2^*$. Moreover, V_i, $i = 1,2$, satisfies Condition 1; V_i and V_i^*, $i = 1,2$, satisfy Condition 2; each pair (V_i, V_i^*), $i = 1,2$, satisfies Conditions 3 and 4. Suppose that $(a,b) \in V_1^*$ and $(b,a) \in V_2^*$. Then, $(a,d) \in V_1$ as the pair (V_1, V_1^*) satisfies Condition 3. But then, $(a,b) \in V_1$

as V_1 satisfies Condition 1, a contradiction. Suppose that $(a, c) \in V_1^*$ and $(c, a) \in V_2^*$. Then, $(c, b) \in V_2$ as the pair (V_2, V_2^*) satisfies Condition 3. But then, $(c, a) \in V_2$ as V_2 satisfies Condition 1, a contradiction. Suppose that $(a, d) \in V_1^*$ and $(d, a) \in V_2^*$. Then, $(a, b) \in V_1$ as the pair (V_1, V_1^*) satisfies Condition 3. But then, $(a, d) \in V_1$ as V_1 satisfies Condition 1, a contradiction. Suppose that $(b, c) \in V_1^*$ and $(c, b) \in V_2^*$. Then, $(b, d) \in V_1$ as the pair (V_1, V_1^*) satisfies Condition 3. But then, $(b, c) \in V_1$ as V_1 satisfies Condition 1, a contradiction. Suppose that $(b, d) \in V_1^*$ and $(d, b) \in V_2^*$. Then, $(b, c) \in V_1$ as the pair (V_1, V_1^*) satisfies Condition 3. But then, $(b, d) \in V_1$ as V_1 satisfies Condition 1, a contradiction. Suppose that $(c, d) \in V_1^*$ and $(d, c) \in V_2^*$. Then, $(d, a) \in V_2$ as the pair (V_2, V_2^*) satisfies Condition 3. But then, $(d, c) \in V_2$ as V_2 satisfies Condition 1, a contradiction. Hence, $V_i^* = \emptyset$, $i = 1, 2$, a contradiction. We have shown that Ω is not strictly decomposable. ■

References

[1] Arrow K.J. (1963), *Social choice and individual values*, Wiley, New York.

[2] Busetto F., Codognato G. (2010), "Nondictatorial Arrovian social welfare functions and integer programs," Working Paper n. 01-10-eco, Dipartimento di Scienze Economiche, Università degli Studi di Udine.

[3] Kalai E., Muller E. (1977), "Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures," *Journal of Economic Theory* **16**, 457-469.

[4] Maskin E. (1979), "Fonctions de préférence collective définies sur des domaines de préférence individuelle soumis à des constraintes," *Cahiers du Séminaire d'Econométrie* **20**, 153-182.

[5] Sethuraman J., Teo C.P., Vohra R.V. (2003), "Integer programming and Arrovian social welfare functions," *Mathematics of Operations Research* **28**, 309-326.

[6] Sethuraman J., Teo C.P., Vohra R.V. (2006), "Anonymous monotonic social welfare functions," *Journal of Economic Theory* **128**, 232-254.