Fiscal Sustainability in a New Keynesian Model

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October 2006
This version, November 2008.

Abstract: Recent work has added government debt and distortionary taxes into New Keynesian models, and analysed optimal fiscal and monetary policy when social welfare is derived from consumers' utility. These papers have shown that, if policy makers can commit to a time inconsistent policy, debt will follow a random walk. In this paper we consider the nature of the time-inconsistency involved and its implication for discretionary policy-making. We show that governments are tempted, given inflationary expectations, to utilise their monetary and fiscal instruments in the initial period to change the ultimate debt burden they need to service. We demonstrate that this temptation is only eliminated if following shocks, the new steady-state debt is equal to the original (efficient) debt level. This implies that under a discretionary policy the random walk result is overturned: debt will always be returned to this initial steady-state even although there is no explicit debt target in the government's objective function. Analytically and in a series of numerical simulations we show which instrument is used to stabilise the debt depends crucially on the degree of nominal inertia and the size of the debt-stock. We also show that the welfare consequences of introducing debt are negligible for precommitment policies, but can be significant for discretionary policy.

JEL Codes: E62, E63

Keywords: New Keynesian Model; Government Debt; Monetary Policy; Fiscal Policy.

*We would like to thank Florin O. Bilbiie, Tatiana Kirsanova, Luisa Lambertini, Ioana Moldovan, Raffaele Rossi, Martin Uribe and participants at seminars at the Universities of Oxford and Valencia for very helpful discussions in the process of drafting this paper. All errors remain our own. We are also grateful to the ESRC, Grant No.RES-156-25-003, for financial assistance. Address for correspondence: Campbell Leith, Department of Economics, University of Glasgow, Adam Smith Building, Glasgow G12 8RT. E-mail c.b.leith@lbss.gla.ac.uk.

1 Overview

Recent work has begun to relax the assumption that all taxes are lump sum in the context of optimal policy in New Keynesian models where social welfare is derived from consumers' utility. Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004) show that in these models optimal debt follows a random walk when policy makers can commit to a time inconsistent policy. In this paper we focus of the nature of the time-inconsistency involved and its implication for discretionary policy-making. This has particular empirical relevance given that policy makers are prepared to place quite tough constraints on their fiscal policies (see for example, the Stability and Growth Pact of EMU) and these constraints are often far from credible.

We verify that the optimal precommitment policy implies a random walk in the steady-state level of debt in the case where both income taxes and government spending are policy instruments. However, our analysis of the time-inconsistency problem reveals that governments are tempted to utilise policy instruments to modify the steady state level of debt in the initial period. This in turn implies that under discretionary policy debt will always be returned to its initial (efficient) steady-state to eliminate this temptation, and debt no longer follows a random walk.

As the literature has not discussed discretionary policy and the time inconsistency problem in this context until now, we analyse it in some detail. For a shock that raises debt, we show that the time inconsistent policy will cut interest rates and government spending in the initial period relative to their new steady-state levels (movements in tax rates are ambiguous), which raises output and inflation. This occurs whether debt is real or nominal. Analytically and using numerical simulations we show that the instrument used to stabilise debt under a discretionary policy depends crucially on the degree of nominal inertia and the size of the debt-stock.

Our numerical simulations also show that the dynamic path of debt following shocks is very different under commitment and discretionary policies, particularly if debt levels are moderately high. Partly as a consequence we show that the welfare consequences of introducing debt without lump sum taxes are relatively small for precommitment policies, but can be significant for discretionary policy. In adding debt to a New Keynesian model, the problem with discretionary policy (relative to commitment) is not that it fails to stabilise the debt stock, but that it is overzealous in doing so.

There has been previous work examining time-consistency problems in the presence of debt, but this has been in models based on flexible prices. In standard felxible price models the policy problem is often trivial. For example, in Lucas and Stokey (1983) they identify three possibilities depending of the initial value of nominal government debt. If debt is positive, the optimal policy is to raise the price level to infinity to deflate the real value of debt without raising distortionary taxes. If the initial level of government liabilities are negative, then monetary policy should be consistent with a price level which ensures the government purchases can be financed without recourse to any distortionary taxation. The "only possibility... of potential practical interest" (Lucas and Stokey, op. cit., page 83) is the case where there is, in the initial period, no outstanding government debt, such that the price level cannot be costlessly manipulated to achieve welfare gains. In this special case, Lucas and Stokey consider a policy mix where monetary policy commits to a path for the price level, and a time consistent policy of tax rates and debt restructuring can be followed. It is only to the extent that the surprise inflation cannot costlessly and instantenously be used to manipulate the real value of the debt stock that a potential time-inconsistency problem exists. However, this time-inconsistency problem need not automatically imply a positive inflation bias in the presence of positive debt stocks, but can be consistent with the Friedman rule (Obstfeld, 1991,1997), or, for alternative preferences, a positive steady-state debt level where the time-inconsistency problem has been eliminated (Ellison and Rankin (2006)). However, given that optimal monetary policy results under flexible prices, such as the Friedman rule, are not robust to the introduction of sticky prices, it is important to extend this analysis to the case of sticky prices and distortionary taxes where surprise inflations cease to be costless.

The plan of the paper is as follows. In Section 2 we outline our model in which consumers supply labour to imperfectly competitive firms who are only able to change prices at random intervals of time. Workers' labour income is taxed. In Section 3 we derive a second-order approximation to welfare for these consumers. This is important since the effective rejection of the Friedman rule in sticky-price models relies on the dominance of the welfare costs of price-distortions relative to the costs reducing the inflation tax. In Section 4, we describe the optimal precommitment policy and analyse the time-inconsistency inherent in that policy, before computing the discretionary policy in Section 5. This then informs the simulation results in section 6, which reveal that operating under discretion overturns the usual random walk result and can potentially generate significant welfare costs.

2 The Model

This section outlines our model. The model is a standard New Keynesian model, but augmented to include the government's budget constraint where government spending is financed by distortionary taxation and or borrowing. This basic set-up is similar to that in Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004) but with some differences. Firstly, we allow the government to vary government spending in the face of shocks in an optimal way, rather than simply treating government spending as an exogenous flow which must be financed. Secondly, we eliminate the usual inflationary bias caused by an inefficiently low level of steady-state output due to imperfect competition and distortionary taxes, by introducing a subsidy financed by lump-sum taxes. However, we do not allow further use of lump-sum taxes to finance government spending and ensure fiscal solvency following shocks - instead governments must adjust spending and/or income taxes to ensure fiscal sustainability. The use of this steady-state subsidy is unavoidable in order to formulate a valid linear-quadratic problem with which to analyse discretionary policy and the nature of the time inconsistency caused by the need to stabilise debt. Benigno and Woodford (*op. cit.*) are able to operate with an inefficient steady-state as they are focussing on a timelessly optimal commitment policy for which it is still possible to formulate a linear-quadratic problem, while Schmitt-Grohe and Uribe (*op. cit.*) utilise second-order solution methods to approximate the Ramsey-planner's policy problem. Neither method can be applied to an analysis of discretionary policy. We examine the households' problem initially, before turning to the firms' problem.

2.1 Households

There are a continuum of households of size one. We shall assume full asset markets, such that, through risk sharing, they will face the same budget constraint. As a result the typical household will seek to maximise the following objective function,

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t, G_t; \xi_t; \xi_t^N) \tag{1}$$

where C,G and N are a consumption aggregate, a public goods aggregate, and labour supply respectively, and ξ is a time preference shock and ξ_t^N is a labour supply shock.

The consumption aggregate is defined as[1]

$$C = (\int_0^1 C(j)^{\frac{\epsilon-1}{\epsilon}} dj)^{\frac{\epsilon}{\epsilon-1}} \tag{2}$$

where j denotes the good's type or variety. The public goods aggregate takes the same form

$$G = (\int_0^1 G(j)^{\frac{\epsilon-1}{\epsilon}} dj)^{\frac{\epsilon}{\epsilon-1}}$$

The elasticity of substitution between varieties $\epsilon_t > 1$ is assumed to time varying as we wish to allow for iid cost-push/mark-up shocks.

The budget constraint at time t is given by

$$\int_0^1 P_t(j) C_t(j) dj + E_t\{Q_{t,t+1} D_{t+1}\} = \Pi_t + D_t + W_t N_t (1 - \tau_t) - T_t$$

where $P_t(j)$ is the price of variety j , D_{t+1} is the nominal payoff of the portfolio held at the end of period t, Π is the representative household's share of profits in the imperfectly competitive firms, W are wages, τ is an wage income tax rate, and T are lump sum taxes. $Q_{t,t+1}$ is the stochastic discount factor for one period ahead nominal payoffs.

Households must first decide how to allocate a given level of expenditure across the various goods that are available. They do so by adjusting the share

[1] We drop the time subscript when all variables in an expression are dated in the same period and there is no possibility of confusion.

of a particular good in their consumption bundle to exploit any relative price differences - this minimises the costs of consumption. Optimisation of expenditure for any individual good implies the demand function given below,

$$C(j) = (\frac{P(j)}{P})^{-\epsilon} C$$

where we have price indices given by

$$P = (\int_0^1 P(j)^{1-\epsilon} dj)^{\frac{1}{1-\epsilon}}$$

The budget constraint can therefore be rewritten as

$$P_t C_t + E_t\{Q_{t,t+1} D_{t+1}\} = D_t + W_t N_t (1 - \tau_t) - T_t \tag{3}$$

where $\int_0^1 P(j)C(j)dj = PC$.

2.1.1 Households' Intertemporal Consumption Problem

The first of the households intertemporal problems involves allocating consumption expenditure across time. For tractability assume that (1) takes the specific form

$$E_0 \sum_{t=0}^{\infty} \beta^t (\frac{C_t^{1-\sigma} \xi_t^{-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma} \xi_t^{-\sigma}}{1-\sigma} - \frac{N_t{}^{1+\varphi} \xi_t^{-\sigma} \xi_t^N}{1+\varphi}) \tag{4}$$

We can then maximise utility subject to the budget constraint (3) to obtain the optimal allocation of consumption across time,

$$\beta (\frac{C_t}{C_{t+1}})^{\sigma} (\frac{\xi_t}{\xi_{t+1}})^{\sigma} (\frac{P_t}{P_{t+1}}) = Q_{t,t+1}$$

Taking conditional expectations on both sides and rearranging gives

$$\beta R_t E_t\{(\frac{C_t}{C_{t+1}})^{\sigma} (\frac{\xi_t}{\xi_{t+1}})^{\sigma} (\frac{P_t}{P_{t+1}})\} = 1 \tag{5}$$

where $R_t = \frac{1}{E_t\{Q_{t,t+1}\}}$ is the gross return on a riskless one period bond paying off a unit of currency in $t+1$. This is the familiar consumption Euler equation which implies that consumers are attempting to smooth consumption over time such that the marginal utility of consumption is equal across periods (after allowing for tilting due to interest rates differing from the households' rate of time preference).

A log-linearised version of (5) can be written as

$$\widehat{C}_t + \widehat{\xi}_t = E_t\{\widehat{C}_{t+1} + \widehat{\xi}_{t+1}\} - \frac{1}{\sigma}(r_t - E_t\{\pi_{t+1}\}) \tag{6}$$

where hatted variables denote percentage deviations from steady-state, $r_t = R_t - \rho$ where $\rho = \frac{1}{\beta} - 1$, and $\pi_t = p_t - p_{t-1}$ is price inflation.

The second foc relates to their labour supply decision and is given by,

$$(1-\tau)\left(\frac{W}{P}\right) = N^{\varphi} C^{\sigma} \xi^{N}$$

Log-linearising implies,

$$-\frac{\overline{\tau}}{1-\overline{\tau}}\widehat{\tau} + \widehat{w} = \varphi\widehat{N} + \sigma\widehat{C} + \widehat{\xi}^{N}$$

2.2 Allocation of Government Spending

The allocation of government spending across goods is determined by minimising total costs, $\int_0^1 P(j)G(j)dj$. Given the form of the basket of public goods this implies,

$$G(j) = (\frac{P(j)}{P})^{-\epsilon} G$$

2.3 Firms

The production function is linear, so for firm j

$$Y(j) = AN(j) \tag{7}$$

where $a = \ln(A)$ is time varying and stochastic. While the demand curve they face is given by,

$$Y(j) = (\frac{P(j)}{P})^{-\epsilon} Y$$

where $Y = \left[\int_0^1 Y(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}$. The objective function of the firm is given by,

$$\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s}\left[P(j)_t Y(j)_{t+s} - W_{t+s}\frac{Y(j)_{t+s}(1-\varkappa)}{A_t}\right] \tag{8}$$

where θ_p is the probability that the firm is unable to change its price in a particular period, and $\varkappa$ is an employment subsidy which can be used to eliminate the steady-state distortion associated with monopolistic competition and distortionary income taxes. Profit maximisation then implies that firms that are able to change price in period t will select the following price,

$$P_t^* = \frac{\sum_{s=0}^{\infty}(\theta_p)^s Q_{t,t+s}\left[\epsilon_t W_{t+s} P_{t+s}^{\epsilon_t}\frac{Y_{t+s}}{A_{t+s}}\right]}{\sum_{s=0}^{\infty}(\theta_p)^s Q_{t,t+s}\left[(\epsilon_t - 1)P_{t+s}^{\epsilon_t} Y_{t+s}(1-\varkappa)\right]}$$

In the working paper version of the paper (Leith and Wren-Lewis (2006)) we deomonstrate that log-linearisation of this pricing behaviour implies a New Keynesian Phillips curve for price inflation which is given by,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(\widehat{mc}_t + \widehat{\mu}_t)$$

where $\gamma = \frac{(1-\theta_p\beta)(1-\theta_p)}{\theta_p}$, $\widehat{mc} = -a + \widehat{w}$ are the real log-linearised marginal costs of production, and $\widehat{\mu}_t = \ln(\frac{\epsilon_t}{\epsilon_t - 1}) - \ln(\frac{\overline{\epsilon}}{\overline{\epsilon}-1})$ is a mark-up shock representing the temporary deviation of the desired markup from its steady-state value.

2.4 Equilibrium

Goods market clearing requires, for each good j,

$$Y(j) = C(j) + G(j) \tag{9}$$

which allows us to write,

$$Y = C + G$$

where aggregate output is defined as, $Y = [\int_0^1 Y(j)^{\frac{\epsilon_t - 1}{\epsilon_t}} dj]^{\frac{\epsilon_t}{\epsilon_t - 1}}$. Log-linearising implies

$$\widehat{Y} = \theta\widehat{C} + (1-\theta)\widehat{G}$$

where we define $\theta = \frac{\overline{C}}{\overline{Y}}$.

2.5 Government Budget Constraint

Combining the series of the representative consumer's flow budget constraints, (3), with borrowing constraints that rule out Ponzi-schemes, gives the intertemporal budget constraint (see Woodford, 2003, chapter 2, page 69),

$$\sum_{T=t}^{\infty} E_t[P_T C_T] \leq D_t + \sum_{T=t}^{\infty} E_t[Q_{t,T}(\Pi_T + W_T N_T(1-\tau_T) - T_T)]$$

Noting the equivalence between factor incomes and national output,

$$PY = WN + \Pi - \varkappa WN$$

and the definition of aggregate demand, we can rewrite the private sector's budget constraint as,

$$D_t = -\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_T G_T - W_T N_T(\tau_T - \varkappa) - T_T)]$$

In order to focus on the time-inconsistency problem associated with the introduction of debt and distortionary taxation to the NNCS model we follow Rotemberg and Woodford (1997) and later authors and introduce a steady-state subsidy. This subsidy offsets, in steady-state, the distortions caused by distortionary taxation and imperfect competition in price setting, and removes the usual desire on the part of policy makers to raise output above its natural level to compensate for these distortions. In other words, this subsidy ensures that the steady state is efficient. The steady state subsidy is financed by lump-sum taxation. We shall assume that both the level of the subsidy and the associated level of lump-sum taxation cannot be altered from this steady state level, so that any changes in the government's budget constraint have to be financed by changes in distortionary taxation or government spending.[2] This implies that $W_T N_T \varkappa = T_T$ in our economy at all points in time, allowing us to simplify the budget constraint to,

$$D_t = -\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_T G_T - W_T N_T \tau_T)]$$

i.e. distortionary taxation and spending adjustments are required to service government debt as well as stabilise the economy. Rewriting in real terms and noting that government debt is dated at the beginning of the period,

$$\frac{B_t}{P_{t-1}}\frac{P_{t-1}}{P_t} = \sum_{T=t}^{\infty} E_t[\beta^{T-t}(\frac{C_t}{C_T})^{\sigma}(\frac{\xi_t}{\xi_T})^{\sigma}(w_T N_T \tau_T - G_T)]$$

where real debt is defined as, $b_t \equiv \frac{B_t}{P_{t-1}}$ and its initial steady-state is given by,

$$\overline{b} = \frac{\overline{w}\overline{N}\overline{\tau} - \overline{G}}{1-\beta}$$

Log-linearising around this steady-state,

$$\begin{aligned} \widehat{b}_t - \pi_t - \sigma(\widehat{C}_t + \widehat{\xi}_t) &= \beta E_t\{\widehat{b}_{t+1} - \pi_{t+1} - \sigma(\widehat{C}_{t+1} + \widehat{\xi}_{t+1})\} \\ &\quad +[-\sigma(1-\beta)(\widehat{C}_t + \widehat{\xi}_t) + \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(\widehat{w}_t + \widehat{N}_t + \widehat{\tau}_t) - \frac{\overline{G}}{\overline{b}}\widehat{G}_t] \end{aligned} \quad (10)$$

Appendix 1 defines the steady-state ratios contained in this log-linearisation as a function of model parameters and the initial steady-state debt-gdp ratio.

3 Optimal policy

In order to derive a welfare function for policy analysis we proceed in the following manner. Firstly, we consider the social planner's problem. We then contrast this with the outcome under flexible prices in order to determine the level of the steady-state subsidy required to ensure the model's initial steady-state is socially optimal. Finally, we construct a quadratic approximation to utility in our sticky-price/distortionary tax economy which assesses the extent to which endogenous variables differ from the efficient equilibrium due to the nominal inertia and tax distortions present in the model. We then recast our model in terms of the 'gap' variables contained within our welfare metric.

[2] The introduction of lump sum taxes in steady state is unavoidable if we are to consider a steady-state with positive levels of government debt. The costs of the subsidy required to offset the impact of distortionary taxation are exactly the same as the revenue generated by the distortionary taxation. Without lump sum taxes we would require negative debt to finance government spending and the subsidy required to remove the monopoly distortion. The subsidy (and the resulting efficient steady state) is in turn required to analyse discretionary policy. Although alternative methods are available to analyse optimal policies under commitment or simple rules, these are not applicable to the analysis of discretionary policy.

3.1 The Social Planner's Problem

The social planner is not constrained by the price mechanism and simply maximises the representative household's utility, (4), subject to the technology, (7), and resource constraints, (9). This yields the following first order conditions,

$$(C_t^*)^{-\sigma} = \chi G_t^{*-\sigma}$$

$$(C_t^*)^{-\sigma} - Y_t^{*\varphi} A_t^{-(1+\varphi)} \xi_t^N = 0$$

where we introduce the '*' superscript to denote the efficient level of that variable. These can be log-linearised around the efficient steady-state, and given the national accounting identity we obtain,

$$\widehat{Y}_t^* = (\frac{1+\varphi}{\sigma+\varphi})a_t - \frac{1}{\sigma+\varphi}\widehat{\xi}_t^N$$

and,

$$\widehat{Y}_t^* = \widehat{C}_t^* = \widehat{G}_t^*$$

3.2 Flexible Price Equilibrium

Appendix 1 derives the subsidy $\varkappa$ required for the flexible price equilibrium to reproduce the efficient steady state. If the government implements its spending plans in line with the social planner's problem in steady-state then the flex price steady-state conditional on the initial fiscal position is the same as the efficient output level. Appendix 1 also defines the steady-state ratios contained in the log-linearised budget constraint, (10), as a function of model parameters and the initial steady-state debt-gdp ratio.

Whether or not the flex price equilibrium in the presence of shocks is the same as the efficient outcome depends on whether or not fiscal variables need to change to satisfy the government's intertemporal budget constraint. With nominal debt and flexible prices, surprise inflation can costlessly deflate the real value of government debt, such that no movement in fiscal instruments are required and the flex price equilibrium will be synonymous with the efficient outcome. However, if debt is real such that fiscal instruments must deviate from their initial steady-state to ensure solvency then the flex price equilibrium need not equal the efficient outcome. However the gaps we employ in the policy problem are actual outcomes relative to efficient outcomes so this potential wedge between the flexible price equilibrium and efficient values of variables need not concern us.

3.3 Social Welfare

Appendix 2 derives the quadratic approximation to utility

$$\Gamma = -\overline{N}^{1+\varphi}\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\{\sigma\theta(\widehat{C}_t - \widehat{C}_t^*)^2 + \sigma(1-\theta)(\widehat{G}_t - \widehat{G}_t^*)^2 + \varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 + \frac{\epsilon_t}{\gamma}\pi_t^2\} + tip + O[2]$$

It contains quadratic terms in price inflation reflecting the costs of price dispersion induced by inflation in the presence of nominal inertia, as well as terms in the consumption, government spending and output gaps i.e. the difference between the actual value of the variable and its optimal value. The weights attached to each element are a function of deep model parameters. The key to obtaining this quadratic specification, suitable for analysing discretionary policy, lies in adopting an employment subsidy which eliminates the steady-state distortions caused by imperfect competition in labour and product markets as well as the steady-state impact of a distortionary income tax. It is important to stress that this subsidy only applies in the steady-state such that it cannot be used as a policy instrument to either stabilise the economy or the government's finances in the face of shocks.

3.4 Gap variables

We have derived welfare based on various gaps, so we now proceed to rewrite our model in terms of the same gap variables to facilitate derivation of optimal policy. The consumption Euler equation can be written in gap form as,

$$(\widehat{C}_t - \widehat{C}_t^*) = E_t\{(\widehat{C}_{t+1} - \widehat{C}_{t+1}^*)\} - \frac{1}{\sigma}((r_t - r_t^*) - E_t\{\pi_{t+1}\})$$

where $r_t^* = \sigma\frac{1+\varphi}{\sigma+\varphi}(E_t\{a_{t+1}\} - a_t) + \sigma(E_t\{\widehat{\xi}_{t+1}\} - \widehat{\xi}_t) - \frac{\sigma}{\sigma+\varphi}(E_t\{\widehat{\xi}_{t+1}^N\} - \widehat{\xi}_t^N)$ is the natural/efficient rate of interest. (This comes from the fact that $\widehat{C}_t^* = \widehat{Y}_t^*$ and the definition of the efficient level of output).

While the NKPC can be written in gap form as,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(\varphi(\widehat{Y}_t - \widehat{Y}_t^*) + \sigma(\widehat{C}_t - \widehat{C}_t^*) + \frac{\overline{\tau}}{1-\overline{\tau}}(\widehat{\tau}_t - \widehat{\tau}_t^*))$$

where, following Benigno and Woodford (2003) we define, $\frac{\overline{\tau}}{1-\overline{\tau}}\ \widehat{\tau}_t^* = \widehat{\mu}_t$. In other words we are defining our 'efficient' tax rate as the tax rate required to perfectly offset the impact of a cost-push shock.[3] If we had access to a lump-sum tax to finance the budget deficit then this would be the optimal tax rate. However, given the need to finance the government liabilities through distortionary taxation, actual tax rates are likely to deviate from the level required to perfectly offset shocks. Appendix 1 rewrites the budget constraint constraint in gap form as,

$$\widehat{b}_t - \pi_t - \sigma(\widehat{C}_t - \widehat{C}_t^*) = \beta\widehat{b}_{t+1} - \beta E_t\{\pi_{t+1} + \sigma(\widehat{C}_{t+1} - \widehat{C}_{t+1}^*)\} + ps_t - f_t - \sigma(1-\beta)(\widehat{C}_t - \widehat{C}_t^*)$$

with the primary surplus defined as,

$$ps_t = \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}[(1+\varphi)(\widehat{Y}_t - \widehat{Y}_t^*) + \frac{1}{1-\overline{\tau}}(\widehat{\tau}_t - \widehat{\tau}_t^*) + \sigma(\widehat{C}_t - \widehat{C}_t^*)] - \frac{\overline{G}}{\overline{b}}(\widehat{G}_t - \widehat{G}_t^*) \quad (11)$$

[3] It should be noted that we could define the tax 'gap' as being the actual tax rate relative to any benchmark tax rate we choose, such as, for example, the initial steady-state tax rate. However, it is convenient to define the gap relative to the tax rate which offsets the impact of a cost-push shock on inflation.

and

$$f_t = -(\sigma(1-\beta\rho_a) + (1-\sigma)(1-\beta))\frac{(1+\varphi)}{\sigma+\varphi}a_t + (\sigma(1-\beta\rho_{\xi^N}) + (1-\sigma)(1-\beta))\frac{\widehat{\xi}_t^N}{\sigma+\varphi} - \frac{\overline{w}\overline{N}}{\overline{b}}\widehat{\mu}_t - \sigma\beta(1-\rho_\xi)\widehat{\xi}_t$$

capturing the extent to which the various shocks hitting our model have fiscal consequences.

4 Precommitment Policy

In this section we shall consider the precommitment policies for our model. The Lagrangian associated with the policy problem under commitment in the presence of a government budget constraint is given by,

$$\begin{aligned} L_t = {} & E_t \sum_{s=0}^{\infty} \beta^s [\sigma\theta(c_{t+s}^g)^2 + \sigma(1-\theta)(g_{t+s}^g)^2 + \varphi(y_{t+s}^g)^2 + \frac{\epsilon}{\gamma}\pi_{t+s}^2 \\ & + \lambda_{t+s}^{\pi}(\pi_{t+s} - \beta\pi_{t+s+1} - \gamma(\varphi y_{t+s}^g + \sigma c_{t+s}^g + \frac{\overline{\tau}}{1-\overline{\tau}}\tau_{t+s}^g) \\ & + \lambda_{t+s}^{y}(y_{t+s}^g - (1-\theta)g_{t+s}^g - \theta c_{t+s}^g) \\ & + \lambda_{t+s}^{b}(\widehat{b}_{t+s} - \pi_{t+s} - \sigma c_{t+s}^g - \beta(\widehat{b}_{t+s+1} - \pi_{t+s+1} - \sigma c_{t+s+1}^g) + \frac{\overline{G}}{\overline{b}} g_{t+s}^g \\ & - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}[(1+\varphi)(y_{t+s}^g) + \frac{1}{1-\overline{\tau}}(\widehat{\tau}_{t+s}^g) + \sigma c_{t+s}^g] + f_{t+s} + \sigma(1-\beta)c_{t+s}^g)] \end{aligned}$$

where λ_{t+s}^{π}, λ_{t+s}^{y}, and λ_{t+s}^{b} are the lagrange multipliers associated with the NKPC, the resource constraint and the government's budget constraint respectively. To simplify notation we have rewritten the gap variables in the form, $x_t^g = \widehat{X}_t - \widehat{X}_t^*$.

4.1 First order conditions for s>0

We shall initially consider the first-order conditions for periods $s > 0$,which are set out in full in Appendix 2.The foc for debt implies that the lagrange-multiplier for debt follows a random walk and this will underpin the random walk for steady-state debt result derived below. Combining this foc with the foc for inflation and the tax rate implies that, in the absence of new information, inflation is zero. Solving the remaining focs implies the following relationships between gapped variables and the lagrange multiplier associated with debt,

$$y_{t+s}^g = -\frac{(\varphi+\sigma) - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda_t^b = -a_1\lambda_t^b$$

for consumption,

$$c^g_{t+s} = -\frac{(\varphi+\sigma)(\frac{\sigma-\theta(1-\theta)}{\theta\sigma}) - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda^b_t = -a_2\lambda^b_t$$

and government spending,

$$g^g_{t+s} = -\frac{(\varphi+\sigma)\frac{\theta}{\sigma} - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda^b_t = -a_3\lambda^b_t$$

The constancy of these various real gaps implies that monetary policy is set such that interest rates are consistent with the natural rate of interest. It is clear from these definitions that the coefficients, $a_i, i = 1, 2, 3$ are positive provided the initial steady-state debt stock satisfies the following conditions,

$$\begin{aligned}(1-\beta)\frac{\overline{B}}{\overline{Y}} &< (\sigma+\varphi)\frac{\theta}{\sigma} \\ (1-\beta)\frac{\overline{B}}{\overline{Y}} &< (\varphi+\sigma)(\frac{\sigma-\theta(1-\theta)}{\theta\sigma}) \\ (1-\beta)\frac{\overline{B}}{\overline{Y}} &< (\sigma+\varphi)\end{aligned} \quad (12)$$

For plausible steady-state debt/GDP ratios all variants of this condition will hold,[4] implying that when $\lambda^b_t > 0$, y^g_{t+s}, c^g_{t+s} and g^g_{t+s} will all be negative, which implies that $\tau^g_{t+s} > 0$.The converse is true when $\lambda^b_t < 0$.

It is helpful to rewrite the definition of the primary surplus, (11), in terms of the value of the lagrange multiplier associated with the government's budget constraint by substituting for the tax rule which applies after the initial period,

$$ps_{t+s} = \frac{\overline{y}}{\overline{b}}[(\frac{\overline{\tau}}{1-\overline{\tau}} - \varphi)y^g_{t+s} - \sigma c^g_{t+s} - (1-\theta)g^g_{t+s}] - f_{t+s}$$

Using the expressions relating the gap variables to the lagrange-multiplier this can be re-written as,

$$ps_{t+s} = \Psi\lambda^b_t - f_{t+s} \quad (13)$$

where $\Psi = (\frac{\overline{B}}{\overline{Y}})^{-1}[(\varphi - \frac{\overline{\tau}}{1-\overline{\tau}})a_1 + \sigma a_2 + (1-\theta)a_3] > 0$, again for debt not too large.[5]

4.2 Commitment Policy and Time-Inconsistency

In this section we consider the case where the policy maker exploits the fact that expectations are given in the initial period. By contrasting the solution in

[4] For the paramater values adopted in the simulation section below, the annualised steady-state debt to GDP ratio would have to exceed 2812.5% for the strongest of these conditions to be violated.

[5] For the parameter values adopted in the simulation section below it is not possible for this coefficient to be negative for any positive debt to gdp ratio.

the initial period to that which follows we can highlight the nature of the time-inconsistency problem facing policy makers, which will help generate intuition for the outcome under discretion.[6] Appendix 3 derives the following expressions, for output

$$y_t^g = (\frac{\sigma\beta}{2(\varphi+\sigma)} - a_1)\widetilde{\lambda}_t^{b.j}$$

consumption,

$$c_t^g = (\frac{(\varphi(1-\theta)+\sigma)\beta}{2(\varphi+\sigma)} - a_2)\widetilde{\lambda}_t^{b.j}$$

and government spending,

$$g_t^g = -(\frac{\varphi\beta}{2(\varphi+\sigma)} + a_3)\widetilde{\lambda}_t^{b.j}$$

where $\widetilde{\lambda}_t^{b,j}$ is the lagrange-multiplier associated with the government's budget constraint under optimal (non-timeless) commitment where $j =$ [real,nom] depending on whether debt is real or nominal.

The initial term in each of these expressions captures the extent to which output and consumption gaps are higher and government spending gaps lower in the initial period as the government attempts to exploit the fact that expectations of the initial period are already formed. Therefore in the face of a shock with negative fiscal consequences we observe higher inflation and relatively higher output and consumption, but lower government spending in the initial period. This implies that there is a negative interest rate gap as monetary policy accommodates debt in the initial period. Whether or not taxes are relatively higher or lower in the initial period is ambiguous. The net effect of these policies is to raise inflation in the initial period. Given this behaviour in the initial period, the initial government surplus is given by,

$$ps_t = (\Psi + \Psi_0)\widetilde{\lambda}_t^{b.j} - f_t$$

where $\Psi_0 = \frac{(\sigma+\varphi)(\theta(1-\theta)-\sigma)+\theta\sigma(1-\beta)\frac{\overline{B}}{\overline{Y}}}{2\frac{\overline{B}}{\overline{Y}}\theta(\sigma+\varphi)} > 0$ provided inequality (12) holds. In other words there is an attempt to reduce the fiscal consequences of the shock in the initial period.

In order to determine the size of the lagrange multiplier associated with the government's IBC, we need to substitute these expressions in the intertemporal budget constraint. This calculation varies according to whether or not debt is real or nominal, since in the later case inflation in the initial period can deflate the real value of the debt. Accordingly in the case of real debt the lagrange multiplier is defined by,

$$\widehat{b}_t - E_{t-1}\pi_t - \sigma\frac{(\varphi(1-\theta)+\sigma)\beta^2}{2(\varphi+\sigma)}\widetilde{\lambda}_t^{b,\text{real}} = (\Psi_0 + \frac{\Psi}{1-\beta})\widetilde{\lambda}_t^{b,\text{real}} - E_t\sum_{s=0}^{\infty}\beta^s f_{t+s}$$

[6] In the working paper version of the paper, Leith and Wren-Lewis (2006), we also derived optimal policy under timeless commitment. However, as our focus is on the nature of the time-inconsistency problem we do not report those results here.

and surprise inflation will not deflate the real value of the debt stock, while in the case of nominal debt we need to take account of the impact of the initial period's inflation on the debt stock,

$$\widehat{b}_t - \pi_t - \sigma\frac{(\varphi(1-\theta)+\sigma)\beta^2}{2(\varphi+\sigma)}\widetilde{\lambda}_t^{b,nom} = (\Psi_0 + \frac{\Psi}{\beta^{-1}-1})\widetilde{\lambda}_t^{b,nom} - E_t\sum_{s=0}^{\infty}\beta^s f_{t+s}$$

Using the expression for the initial rate of inflation and solving for the lagrange-multiplier for the nominal debt case yields,

$$(\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma + \Psi_0 + \frac{\Psi}{1-\beta} + \sigma\frac{(\varphi(1-\theta)+\sigma)\beta^2}{2(\varphi+\sigma)})\widetilde{\lambda}_t^{b,nom} = \widehat{b}_t + E_t\sum_{s=0}^{\infty}\beta^s f_{t+s}$$

In other words the lagrange-multiplier is proportional to the sum of the initial debt-disequilibrium and the expected discounted value of the fiscal effects of shocks. However, cet. par. the value of the multiplier will not be as large when the policy maker exploits fixed expectations in the initial period to raise additional tax revenue and deflate the debt. This implies that, in the case of a shock with a higher debt stock, output, consumption and government spending will not fall by as much, and taxes will not need to rise by as much to support the new steady-state debt stock, which is lower than it would be under a policy which did not exploit the fact that expectations are given in the initial period. This allows us to state our first proposition.

Proposition 1 *Under commitment there is an attempt in the first period to reduce debt following a positive fiscal shock*
Proof. *The new steady-state debt stock under (non-timeless) commitment for the case of nominal and real debt are given by,*

$$\overline{b}^{C,nom} = (\frac{\Psi}{1-\beta})\widetilde{\lambda}_t^{b,nom} < \overline{b}^{C,\mathrm{real}} = (\frac{\Psi}{1-\beta})\widetilde{\lambda}_t^{b,\mathrm{real}} < \widehat{b}_t + E_t\sum_{s=0}^{\infty}\beta^s f_{t+s}$$

These inequalities, together with the analysis above, show that some of the fiscal consequences of the shocks will be undone by the policy implemented in the first period. ■

It is important to note that if there was no attempt to behave differently in the initial period then there would be full accommodation of all the fiscal consequences of shocks. The scale of debt reduction in the initial period will be greater in the case of nominal debt than real debt since debt deflation is possible. It is also interesting to note from (15) in Appendix 2 that inflation in the initial period is lower when debt is nominal as the debt deflation in the initial period reduces the need to adjust other instruments and thereby mitigates their inflationary consequences.

We are now in a position to fully describe the response to shocks under commitment. Shocks only have an effect on welfare-relevant gap variables to the extent that they have fiscal repercussions, the financing of which limits the extent to which monetary and fiscal policy can achieve the first-best solution. Under both forms of commitment inflation beyond the initial period is always zero. Outside of the initial period, policy allows the fiscal effects of shocks to be fully reflected in the debt stock and to only adjust fiscal instruments (government spending gaps and the tax gap) to the extent required to support the new steady-state debt stock. Time inconsistent (non-timeless) optimal commitment policy improves welfare further by exploiting the Phillips curve to run an accommodative monetary policy in the initial period along with an increase in the primary surplus (although this need not imply higher tax rates) which generates inflation in the initial period, but thereafter it also adjusts fiscal instruments only to the extent required to support the new steady-state debt stock. The generation of inflation in the initial period also means that it matters whether debt is real or nominal: the increase in steady state debt (and associated changes in other variables) will be less when debt is nominal. In both cases, if shocks raise debt, then steady state taxes are higher, and steady state government spending, private consumption and output are all lower. Debt will slowly evolve until it reaches a new steady-state value consistent with the higher taxes, lower government spending and reduced consumption and output.

5 Discretionary Policy

We now examine the discretionary solution to the problem. Appendix 3 shows that the Bellman equation for this problem can be written as,

$$V(\mathbf{S}_{t-1};\boldsymbol{\xi}_t) = \underset{\mathbf{u}_t}{Min}(\boldsymbol{\pi}_t\mathbf{R}\boldsymbol{\pi}_t + \mathbf{u}_t'\mathbf{Q}\mathbf{u}_t) + \beta E_t V(\mathbf{S}_t;\boldsymbol{\xi}_{t+1}) \tag{14}$$

subject to,

$$\boldsymbol{\pi}_t = \mathbf{C1}\mathbf{S}_{t-1} + \mathbf{C2}\mathbf{u}_t + \mathbf{C3}\boldsymbol{\xi}_t$$

and,

$$\mathbf{S}_t = \mathbf{D1}\mathbf{S}_{t-1} + \mathbf{D2}\mathbf{u}_t + \mathbf{D3}\boldsymbol{\xi}_t$$

where **C1**, **C2**, **C3**, **D1**, **D2** and **D3** are coefficient matrices defined in Appendix 4 after exploiting the linear-quadratic form of the problem to eliminate expectations. $\mathbf{R} = \begin{bmatrix} \frac{\varepsilon}{\gamma} \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} \theta(\sigma+\varphi\theta) & 0 & 2(1-\theta)\theta\varphi \\ 0 & 0 & 0 \\ 0 & 0 & (1-\theta)(\sigma+\varphi(1-\theta)) \end{bmatrix}$, and $\mathbf{S}_{t-1} = \begin{bmatrix} \widehat{b}_t \\ a_{t-1} \\ \widehat{\xi}^N_{t-1} \\ \mu_{t-1} \\ \widehat{\xi}_{t-1} \end{bmatrix}$ and $\mathbf{u}_t = \begin{bmatrix} c^g_t \\ \tau^g_t \\ g^g_t \end{bmatrix}$ are the vectors of state and control variables respectively, while $\boldsymbol{\xi}_t$ is a vector of iid innovations to the model's shock processes.

The solution to this problem is given in Appendix 3. Unfortunately it is too unwieldy to yield any real intuition. Nevertheless, after imposing the solved value for the undetermined coefficients used to re-cast the problem as a recursive one, we can examine the evolution of the state variables under the optimal discretionary policy,

$$\mathbf{S}_t = [\mathbf{D1} - \mathbf{D2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2}]\mathbf{S}_{t-1} + [\mathbf{D3} - [\mathbf{U1}]^{-1}\mathbf{U3}]\boldsymbol{\xi}_{\mathbf{t}} \equiv \mathbf{G}\mathbf{S}_{t-1} + \mathbf{H}\boldsymbol{\xi}_t$$

where **U1**, **U2** and **U3** are defined in Appendix 4. In order for steady-state debt to follow a random walk under discretion in the face of nonpermanent shocks, the element $G_{1,1}$ must equal 1. This allows us to state our second proposition

Proposition 2 *Under discretion, debt will in general no longer follow a random walk*
Proof. *By counterexample - substitution of the central parameter set utilised in the simulation section below, and a large number of variants of that set, imply that* $|G_{1,1}| < 1$. ■

In other words, under a discretionary policy debt eventually returns to its efficient solution, which given subsidies is equal to its pre-shock level. Only by chance (and not for any parameter values close to standard calibrations) will steady-state debt follow the random walk property that was inevitable under the commitment case.[7]

It is of some interest to investigate how debt is returned to its original level under a discretionary policy. Equation (19) in Appendix 3 shows that there is a linear relationship between the consumption gap and inflation and between the government spending gap and inflation under discretion,

$$\begin{aligned} g_t^g &= -\frac{(\theta(\varphi+\sigma) + \sigma(\beta(1+\varphi)-1)\frac{\overline{B}}{\overline{Y}})\varepsilon}{\sigma(\varphi+\sigma)((1-\theta) + (1-\beta)\frac{\overline{B}}{\overline{Y}} + 1 + \frac{\overline{B}}{\overline{Y}}\gamma)}\pi_t \\ c_t^g &= -\frac{(\sigma(\sigma+\varphi - \frac{\sigma+\varphi}{\sigma}\theta(1-\theta)) - \frac{\overline{B}}{\overline{Y}}\sigma(\sigma\beta + \theta(1-\beta) + \varphi\beta(1-\theta)))\varepsilon}{(\varphi+\sigma)\sigma\theta((1-\theta) + (1-\beta)\frac{\overline{B}}{\overline{Y}} + 1 + \frac{\overline{B}}{\overline{Y}}\gamma)}\pi_t \end{aligned}$$

. Combining this with the definition of output allows us to obtain the following relationship between the output gap and inflation under discretion,

$$y_t^g = -\frac{((\varphi+\sigma) - (1+\beta(\sigma-1))\frac{\overline{B}}{\overline{Y}})\varepsilon}{(\varphi+\sigma)((1-\theta) + (1-\beta)\frac{\overline{B}}{\overline{Y}} + 1 + \frac{\overline{B}}{\overline{Y}}\gamma)}\pi_t$$

[7] By construction, our original steady state is also the efficient steady state. We can therefore describe the discretionary solution as returning to either to its original level or the efficient level. We choose the latter for the following reason. If we allow our original steady state to involve small distortions (of the kind examined in Woodford (2003, chapter 6)), then the discretionary solution moves debt away from this initial steady state to a level that eliminates this distortion. However, to demonstrate this involves considerable additional analysis, adds no further insight to the propositions in this paper and pushes us away from the region in which our linear-quadratic formulation of discretionary policy remains valid, which is why we focus on an efficient steady state in this paper. This additional analysis is available upon request.

For plausible labour supply parameters, the government spending gap has the opposite sign to the rate of inflation under discretion, but the signs of the other relationships depend crucially on the size of the debt stock. For small steady-state debt/GDP ratios under discretion the output gap and consumption will have the opposite sign to the rate of inflation. Basically, following a shock with negative fiscal consequences, government spending will fall and taxes will rise in order to stabilise the debt stock. As tax rates are an element of marginal costs, the tax rise will fuel inflation. Monetary policy will be tightened in order to control this inflation, and this will serve to reduce consumption (and output). Although the tightening of monetary policy will raise debt service costs, the relatively small size of the initial debt stock ensures that this is not a significant problem.

However, for sufficiently large debt stocks policy-makers must recognise the negative effect of monetary policy on debt service costs. This raises the efficacy of using monetary policy to stabilise the debt such that, with a sufficiently large steady-state debt/gdp ratio the output and consumption gaps will move in the same direction as inflation. In other words, despite the fact that government spending falls, consumption (and output) will increase as a result of a relaxation of monetary policy needed to stabilise debt. Whether or not taxes increase or decrease, augmenting or offsetting this inflationary impulse, is ambiguous, but as the steady-state debt/gdp ratio rises the inflationary consequences of taxation becomes more significant (as at higher levels of steady-state taxation a marginal increase in tax rates becomes a greater drag on labour supply) such that taxes are less likely to be used as a tool to stabilise the debt, and are more likely to be utilised to control inflation.[8]

This allows us to state out third and final proposition.

Proposition 3 *Once the initial steady-state debt-gdp ratio exceeds some critical value, interest rates will be set in order to return debt to its efficient level*
Proof. *The level of debt at which monetary policy moves from an anti-inflationary stance to one of fiscal accommodation is given by,*

$$\frac{\overline{B}}{\overline{\overline{Y}}} > \frac{(\sigma - \theta(1-\theta))(\sigma + \varphi)}{\sigma\beta + \beta(1-\theta)\varphi + (1-\beta)\theta}$$

■

For the parameter values considered below this critical value occurs at an annualised debt/gdp ratio of only 30.4%[9] Given the simple linear relationship between output and inflation under discretion it is also possible to assess the

[8]The changing balance between fiscal and monetary stabilisation of debt under discretion has echoes of the policies observed under the alternative determinate combinations of simple monetary and fiscal policy rules (see, for example, Leeper (1991) and Leith and Wren-Lewis (2000,2006)).

[9]A numerical analysis of the contributions of various policy instruments to debt stabilisation is conducted in the following section.

relative volatility of the output gap and inflation in the face of shocks. The relative size of the variances will be given by,

$$\frac{Var(y^g)}{Var(\pi)} = \left(\frac{((\varphi+\sigma) - (1+\beta(\sigma-1))\frac{\overline{B}}{\overline{Y}})\varepsilon}{(\varphi+\sigma)((1-\theta)+(1-\beta)\frac{\overline{B}}{\overline{Y}}+1+\frac{\overline{B}}{\overline{Y}}\gamma)} \right)^2$$

This implies that the volatility of inflation relative to output increases in the the steady-state debt-GDP ratio. It is also the case that raising the degree of price flexibility (raising γ) will reduce the adjustment of output and government spending relative to inflation in responding to shocks under discretion, and in the limit as $\gamma \to \infty$ and prices become flexible, all adjustment is through surprise inflation deflating the nominal debt stock. These results are confirmed in the numerical analysis below.

6 Optimal Policy Simulations

In this section we outline the response of the model to a series of shocks, and illustrate the three propositions established above. Following the econometric estimates in Leith and Malley (2005) we adopt the following parameter set, $\varphi = 1$, $\sigma = 2, \mu = 1.2$, $\overline{\epsilon} = 6$, $\beta = 0.99$, and, following Gali (1994) the share of government consumption in GDP, $1-\theta = 0.25$. In our benchmark simulations we assume a degree of price stickiness of $\theta_p = 0.75$, which implies that an average contract length of one year, and an initial debt-GDP ratio of 60%. However, we also explore the implications of alternative assumptions regarding the degree of price stickiness and the initial steady-state debt stock. The productivity shock follows the following pattern,

$$a_t = \rho_a a_{t-1} + \xi_t$$

where we adopt a degree of persistence in the productivity shock of $\rho_a = 0.99$. We assume a similar dynamic structure for the labour supply and taste shocks, consistent with the evidence for both forms of shock in Smets and Wouters (2005). Ireland (2004) finds similar persistence in the productivity shock. Smets and Wouters (2005) assume that cost push shocks are iid in nature. We do the same, but if we allow for persistent cost-push shocks as in Ireland (2004) this will raise the welfare costs for this particular shock relative to the numbers we report below.

6.1 Debt under commitment and discretion

We begin by confirming our second proposition: the absence of the random walk result in debt under discretion. Figure 1 plots the coefficient on the lagged value of debt in the state-space solution under discretion, $G_{1,1}$ as a function of the degree of price stickiness and steady-state debt-gdp ratio. This confirms that our solution is stationary since $|G_{1,1}| < 1$. It is only at very low levels of the debt-gdp ratio that this coefficient tends to one. The reason is implicit in the analysis above and will be explored in the following section. In essence, at very low debt-gdp ratios surprise inflation and monetary policy accommodation are less effective in stabilising debt so the temptation to use them weakens. Instead, adjustment takes place through spending and taxation which suffer less from the time-inconsistency problem, allowing a more gradual stabilisation of debt. As we increase the size of the debt-stock, we find that at relatively modest debt/GDP ratios the sign on the lagged debt stock in the discretionary solution switches from being positive to negative. This tends to imply greater volatility in the various gap variables in face of shocks, with negative implications for welfare that we note below.

We now compare how policy makers stabilise the economy over time under both commitment and discretion. Figure 2 details the paths of key endogenous variables following a persistent technology shock under these two policy regimes. Commitment allows policy makers to exploit the fact that expectations are given in the initial period, and we observe a (small) initial cut in the government spending and interest rate gaps. This is an example of our first proposition, that in the first period the commitment policy moves to reduce the steady state increase in debt. This fuels inflation despite the moderating effects of a fall in taxation. As a result, debt actually falls in the initial period, which allows lower levels of taxation and higher levels of output, government spending and consumption than would otherwise be the case beyond the initial period.

Under discretion we observe that debt returns to its initial (=efficient) level. This requires a more substantial response for all variables. Our third and final proposition noted that the direction of response of variables depends crucially on the steady-state debt-gdp ratio. In our simulations our chosen debt/gdp ratio of 60% constitutes a 'high' level of debt when considering the analytical results derived above. This implies that, initially, interest rates and government spending are cut. The cut in interest rates boosts consumption by enough to offset the fall in government spending, such that overall output increases. On top of this taxes are also initially increased and this further fuels inflation in the initial period. This serves to reduce the debt stock in the initial period. Since under discretion governments are performing a period-by-period optimisation, this drop in debt results in incentives to move policy instruments in the opposite direction in the following period, but with the same basic pattern outlined in the section on discretion above. Instruments then follow a damped cycle until the debt stock has returned to its initial value.[10]

Similar responses emerge for other shocks present in the model since it is only to the extent that shocks have fiscal consequences that we cannot use the mixture of fiscal and monetary policy instruments to offset the impact of shocks on gap variables. These are summarised in Table 1.

[10] An inefficient initial steady-state would create an additional incentive to drive debt to a lower level consistent with efficiency, but the burden of adjustment amongst instruments under discretion would be similar to that identified here.

Table 1 - Welfare Consequences of Shocks under Alternative Policies.[11]

	Discretion	Commitment
Technology	0.3645	0.0085
Labour Supply	0.0910	0.0021
Mark-Up (iid)	0.1032	0.0024
Taste	0.3620	0.0085

The main point to take from Table 1 is the relative magnitudes of the welfare effects of the shock under discretion relative to commitment, which are significantly higher under discretion. Under commitment there is only a very partial offsetting of the fiscal consequences of the shock - outside of the initial period instruments are adjusted to support the new debt stock that emerges as a result of the shock. In contrast policy under discretion completely offsets the fiscal consequences of shocks, which requires greater short-term movement in policy instruments which then generates the welfare effects captured in Table 1.

6.2 How debt is controlled under discretion.

In light of these results it is informative to know which policy instruments bear the brunt of the adjustment. To do this we calculate the contribution of tax revenues, government spending, surprise inflation and lower debt service costs (but excluding surprise inflation) to returning debt to its steady-state value following a negative technology shock. In the graphs below, the Z axis captures the proportion of the fiscal consequences of the shock offset by the mechanism considered in that graph. These results are plotted as a function of the steady-state debt-gdp ratio and the degree of price stickiness, θ_P.

The first row captures the contribution of extra tax revenues and reduced government spending to debt stabilisation. Here we can see that it is only at very low debt levels and high degrees of price stickiness that increased tax revenues play a significant role in returning debt to its steady-state value. Reductions in government expenditure do not contribute to fiscal stabilisation except at moderate degrees of price stickiness and relatively low levels of steady-state debt. With fiscal instruments making little contribution to debt stabilisation under discretion for plausible levels of debt and price stickiness, the burden of adjustment must fall on monetary policy. The is confirmed in the second row of graphs in Figure 3. Here we find that beyond debt-gdp ratios of 30% monetary policy reduces debt service costs in an attempt to return the debt to its initial value. As the debt to gdp ratio rises the efficacy of stabilising debt in this way increases and it almost single-handedly offsets the fiscal consequences of the shock as debt-gdp ratios rise to 200%. As prices become more flexible, the ability of the monetary authorities to engineer a change in real interest rates is reduced and there is greater reliance on surprise inflation to return debt to its steady-state. Interestingly, the use of reduced debt service costs eventually

[11] Following Erceg *et al* (2000) the costs are expressed as a proportion of one period's steady-state consumption divided by the variance of the shock innovation.

declines as prices become increasingly sticky. However this reflects the fact that at high levels of nominal inertia a relatively modest decline in real interest rates produces a significant rise in the tax base which boosts tax revenues (this is evident from the increasing importance of tax revenues at high levels of price stickiness).

For our central parameter set with price contracts lasting for 1 year ($\theta_p = 0.75$) and an annualised debt to gdp ratio of 60%, the relative contributions of taxation and government spending to the stabilisation of debt are 18.38% and 2.93%, respectively. Surprise inflation accounts for 7.53% of the required stabilisation of debt. However by far the greatest mechanism for stabilising the debt comes from reduced debt service costs which accounts of 78.69% of the adjustment.

7 Conclusions

In this paper we examined the ability of government spending, distortionary taxes and monetary policy to offset the effects of various shocks when policymakers did not have access to lump-sum taxes to balance the budget. In a simple New Keynesian framework, the three instruments available to policy makers could not offset the impact of shocks on welfare relevant gap variables since the shocks also had an impact on the government's budget constraint. We analytically derived commitment policy and confirmed that debt would follow a random walk. Outside the initial period tax and government spending variables would only adjust to maintain this new steady-state level of debt. Inflation would be zero and monetary policy would ensure that nominal interest rates were consistent with the natural rate of interest.

However, in the initial period under optimal commitment, policy makers exploit the fact that expectations are given in the initial period. This reveals the nature of the time-inconsistency problem inherent in the commitment solution, whereby the fiscal authorities have the incentive, given expectations, to use fiscal policy more aggressively and monetary policy more accommodatingly in the initial period to reduce the subsequent debt-disequilibrium and the costs associated with sustaining a given debt level. In the case of a shock with negative fiscal consequences this will imply that the fiscal authorities cut government spending in the initial period while the monetary authorities cut interest rates. The net impact of these policies are to increase the primary surplus and raise marginal costs and fuel inflation in the initial period. This slows the initial rise in the debt stock allowing the new steady-state debt stock to be supported with lower permanent increases in tax rates, and falls in consumption, output and government spending.

We then turn to the discretionary solution, where governments follow a time consistent policy. The random walk result no longer holds, and instead debt gradually returns to its steady state level. Only by returning debt to its steady state can the incentive to reduce debt in the initial period noted under commitment be eliminated. Analytically, we demonstrated that exactly how this is achieved depends crucially on the debt-gdp level and the degree of price stickiness. Through simulations we demonstrated that for plausible parameter values a negative fiscal shock is likely to be unwound by monetary policy engineering a reduction in debt service costs. We have also shown that the welfare consequences of shocks to debt when policy operates under discretion can be significant, which in turn implies that welfare analysis under discretion using models that employ the fiction of lump sum taxes may be incomplete.

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Appendix 1 - Optimal Policy

(1) Flexible Price Equilibrium

Profit-maximising behaviour implies that firms will operate at the point at which marginal costs equal marginal revenues,

$$\begin{aligned} -\ln(\mu_t) &= mc_t \\ \left(1 - \frac{1}{\epsilon_t}\right) &= \frac{(1-\varkappa)}{(1-\tau_t)}(N_t^n)^{(\varphi)} A_t^{-1} (C_t^n)^{\sigma} \xi_t^N \end{aligned}$$

In the initial steady-state this reduces to,

$$\left(1 - \frac{1}{\overline{\epsilon}}\right) = \frac{(1-\varkappa)}{(1-\overline{\tau})}(\overline{N}^n)^{\varphi}(\overline{C}^n)^{\sigma}$$

If the subsidy $\varkappa$ is given by

$$(1-\varkappa) = (1 - \frac{1}{\overline{\epsilon}})(1-\overline{\tau})$$

then

$$(\overline{C}^n)^{-\sigma} = (\overline{N}^n)^{\varphi}$$

which is identical to the optimal level of employment in the efficient steady-state. Given the steady-state government spending rule,

$$\frac{\overline{G}}{\overline{Y}} = (1 + \chi^{-\frac{1}{\sigma}})^{-1}$$

the steady-state level of output is given by,

$$\overline{Y} = \overline{N} = (1 + \chi^{\frac{1}{\sigma}})^{\frac{\sigma}{\sigma+\varphi}}$$

and, if the subsidy is in place, then the steady-state real wage is given by,

$$\overline{w} = \frac{1}{1-\overline{\tau}}$$

The steady-state tax rate required to support a given debt to GDP ratio is given by,

$$\overline{\tau} = \frac{(1-\beta)\frac{\overline{B}}{\overline{Y}} + \frac{\overline{G}}{\overline{Y}}}{1 + (1-\beta)\frac{\overline{B}}{\overline{Y}} + \frac{\overline{G}}{\overline{Y}}}$$

and the tax revenues relative to debt this implies are given by,

$$\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}} = \frac{\frac{\overline{\tau}}{1-\overline{\tau}}}{\frac{\overline{B}}{\overline{Y}}}$$

This is enough to define all log-linearised relationships dependent on model parameters and the initial debt to gdp ratio.

(2) Derivation of Welfare

Individual utility in period t is

$$\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}+\chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}-\frac{N_t{}^{1+\varphi}\xi_t^{-\sigma}\xi_t^{N}}{1+\varphi}$$

Before considering the elements of the utility function we need to note the following general result relating to second order approximations,

$$\frac{Y_t-Y}{Y_t}=\widehat{Y}_t+\frac{1}{2}\widehat{Y}_t^2+O[2]$$

where $\widehat{Y}_t=\ln(\frac{Y_t}{Y})$, $O[2]$ represents terms that are of order higher than 2 in the bound on the amplitude of the relevant shocks. This will be used in various places in the derivation of welfare. Now consider the second order approximation to the first term,

$$\begin{aligned}\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} &= \overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})-\frac{\sigma}{2}\overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})^2\\ &\quad -\sigma\overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})(\xi_t-1)+tip+O[2]\end{aligned}$$

where tip represents 'terms independent of policy'. Using the results above this can be rewritten in terms of hatted variables,

$$\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}=\overline{C}^{1-\sigma}\{\widehat{C}_t+\frac{1}{2}(1-\sigma)\widehat{C}_t^2-\sigma\widehat{C}_t\widehat{\xi}_t\}+tip+O[2]$$

Similarly for the term in government spending,

$$\chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}=\chi\overline{G}^{1-\sigma}\{\widehat{G}_t+\frac{1}{2}(1-\sigma)\widehat{G}_t^2-\sigma\widehat{G}_t\widehat{\xi}_t\}+tip+O[2]$$

The final term in labour supply can be written as,

$$\frac{N_t^{1+\varphi}\xi_t^N\xi_t^{-\sigma}}{1+\varphi}=\overline{N}^{1+\varphi}\{\widehat{N}_t+\frac{1}{2}(1+\varphi)\widehat{N}_t^2-\sigma\widehat{N}_t\widehat{\xi}_t+\widehat{N}_t\widehat{\xi}_t^N\}+tip+O[2]$$

Now we need to relate the labour input to output and a measure of price dispersion. Aggregating the individual firms' demand for labour yields,

$$N=(\frac{Y}{A})\int_0^1(\frac{P_H(i)}{P_H})^{-\epsilon_t}di$$

It can be shown (see Woodford, 2003, Chapter 6) that

$$\begin{aligned}
\widehat{N} &= \widehat{Y} - a + \ln[\int_0^1 (\frac{P(i)}{P})^{-\epsilon_t} di] \\
&= \widehat{Y} - a + \frac{\epsilon}{2} var_i\{p(i)\} + O[2]
\end{aligned}$$

so we can write

$$\begin{aligned}
\frac{N_t^{1+\varphi}\xi_t^N \xi_t^{-\sigma}}{1+\varphi} &= \overline{N}^{1+\varphi}\{\widehat{Y}_t + \frac{1}{2}(1+\varphi)\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t + \widehat{Y}_t\widehat{\xi}_t^N - \sigma\widehat{Y}_t\widehat{\xi}_t + \frac{\epsilon}{2} var_i\{p_t(i)\}\} \\
&\quad + tip + O[2]
\end{aligned}$$

Using these expansions, individual utility can be written as

$$\begin{aligned}
\Gamma_t &= \overline{C}^{1-\sigma}\{\widehat{C}_t + \frac{1}{2}(1-\sigma)\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} \\
&\quad + \chi\overline{G}^{1-\sigma}\{\widehat{G}_t + \frac{1}{2}(1-\sigma)\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} \\
&\quad - \overline{N}^{1+\varphi}\{\widehat{Y}_t + \frac{1}{2}(1+\varphi)\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t \\
&\quad - \sigma\widehat{Y}_t\widehat{\xi}_t + \widehat{Y}_t\widehat{\xi}_t^N + \frac{\epsilon}{2} var_i\{p_t(i)\}\} \\
&\quad + tip + O[2]
\end{aligned}$$

Using second order approximation to the national accounting identity,

$$\theta\widehat{C}_t = \widehat{Y}_t - (1-\theta)\widehat{G}_t - \frac{1}{2}\theta\widehat{C}_t^2 - \frac{1}{2}(1-\theta)\widehat{G}_t^2 + \frac{1}{2}\widehat{Y}_t^2 + O[2]$$

With the steady-state subsidy in place and government spending chosen optimally, the following conditions hold in the initial steady-state, $\overline{C}^{1-\sigma}$

$$\overline{C}^{1-\sigma} = \overline{N}^{1+\varphi}\theta$$

and,

$$\chi\overline{G}^{1-\sigma} = \overline{N}^{1+\varphi}(1-\theta)$$

Which allows us to eliminate the levels terms and rewrite welfare as,

$$\begin{aligned}
\Gamma_t &= \overline{C}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} + \chi\overline{G}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} \\
&\quad - \overline{N}^{1+\varphi}\{\frac{1}{2}\varphi\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t - \sigma\widehat{Y}_t\widehat{\xi}_t + \widehat{Y}_t\widehat{\xi}_t^N + \frac{\epsilon}{2} var_i\{p_t(i)\}\} \\
&\quad + tip + O[2]
\end{aligned}$$

We now need to rewrite this in gap form using the focs for the social planner to eliminate the term in the technology shock,

$$\begin{aligned}
\Gamma_t &= -\overline{N}^{1+\varphi}\frac{1}{2}\{\sigma\theta(\widehat{C}_t - \widehat{C}_t^*)^2 + \sigma(1-\theta)(\widehat{G}_t - \widehat{G}_t^*)^2 + \varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 + \epsilon var_i\{p_t(i)\}\} \\
&\quad + tip + O[2]
\end{aligned}$$

Using the result from Woodford (2003) that

$$\sum_{t=0}^{\infty}\beta^t var_i\{p_t(i)\} = \frac{\theta}{(1-\theta)(1-\theta\beta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2 + t.i.p + O[2]$$

we can write the discounted sum of utility as,

$$\Gamma = -\overline{N}^{1+\varphi}\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\{\sigma\theta(\widehat{C}_t-\widehat{C}_t^*)^2 \quad +\sigma(1-\theta)(\widehat{G}_t-\widehat{G}_t^*)^2+\varphi(\widehat{Y}_t-\widehat{Y}_t^*)^2+\frac{\epsilon}{\gamma}\pi_t^2\}$$
$$+tip+O[2]$$

(3) The budget constraint using gap variables

The log-linearised budget constraint is given by,

$$\begin{aligned}\widehat{b}_t-\pi_t-\sigma(\widehat{C}_t+\widehat{\xi}_t) &= \beta\widehat{b}_{t+1}-\beta E_t\{\pi_{t+1}+\sigma(\widehat{C}_{t+1}+\widehat{\xi}_{t+1})\} \\ &\quad +\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(\widehat{w}_t+\widehat{N}_t+\widehat{\tau}_t)-\frac{\overline{G}}{\overline{b}}\widehat{G}_t-\sigma(1-\beta)(\widehat{C}_t+\widehat{\xi}_t)\end{aligned}$$

Using the labour supply function to eliminate real wages and the definition of efficient output to eliminate the technology shock,

$$\begin{aligned}\widehat{b}_t-\pi_t-\sigma(\widehat{C}_t+\widehat{\xi}_t) &= \beta\widehat{b}_{t+1}-\beta E_t\{\pi_{t+1}+\sigma(\widehat{C}_{t+1}+\widehat{\xi}_{t+1})\} \\ &\quad -\sigma(1-\beta)(\widehat{C}_t+\widehat{\xi}_t)-\frac{\overline{G}}{\overline{b}}\widehat{G}_t \\ &\quad +\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t-\widehat{Y}_t^*)+\frac{1}{1-\overline{\tau}}\widehat{\tau}_t+\sigma(\widehat{C}_t-\widehat{C}_t^*)+\widehat{Y}_t^*)\end{aligned}$$

Gapping the remaining variables and combining shock terms,

$$\begin{aligned}\widehat{b}_t-\pi_t-\sigma(\widehat{C}_t-\widehat{C}_t^*) &= \beta\widehat{b}_{t+1}-\beta E_t\{\pi_{t+1}+\sigma(\widehat{C}_{t+1}-\widehat{C}_{t+1}^*)\}-f_t \\ &\quad -\sigma(1-\beta)(\widehat{C}_t-\widehat{C}_t^*)-\frac{\overline{G}}{\overline{b}}(\widehat{G}_t-\widehat{G}_t^*) \\ &\quad +\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t-\widehat{Y}_t^*)+\frac{1}{1-\overline{\tau}}(\widehat{\tau}_t-\widehat{\tau}_t^*)+\sigma(\widehat{C}_t-\widehat{C}_t^*))\end{aligned}$$

where

$$\begin{aligned}f_t &= -(\sigma(1-\beta\rho_a)+(1-\sigma)(1-\beta))\frac{(1+\varphi)}{\sigma+\varphi}a_t+(\sigma(1-\beta\rho_{\xi^N}) \\ &\quad +(1-\sigma)(1-\beta))\frac{\widehat{\xi}_t^N}{\sigma+\varphi}-\frac{\overline{w}\overline{N}}{\overline{b}}\mu_t-\sigma\beta(1-\rho_\xi)\widehat{\xi}_t\end{aligned}$$

captures the fiscal consequences of the various shocks hitting the economy.

Appendix 2 - Optimal Commitment Policy

(1) First order conditions for s>0

The first-order conditions from optimisation for periods $s > 0$ are given by the following set of equations. Firstly for consumption,

$$2\sigma\theta c^g_{t+s} - \gamma\sigma\lambda^\pi_{t+s} - \theta\lambda^y_{t+s} - \sigma\lambda^b_{t+s} - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}\sigma\lambda^b_{t+s} + \sigma(1-\beta)\lambda^b_{t+s} + \beta\sigma\lambda^b_{t+s-1} = 0$$

government spending,

$$2\sigma(1-\theta)g^g_{t+s} - (1-\theta)\lambda^y_{t+s} + \frac{\overline{G}}{\overline{b}}\lambda^b_{t+s} = 0$$

the output gap,

$$2\varphi y^g_{t+s} - \gamma\varphi\lambda^\pi_{t+s} + \lambda^y_{t+s} - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(1+\varphi)\lambda^b_{t+s} = 0$$

debt,

$$E_t\lambda^b_{t+s} - \lambda^b_t = 0$$

inflation,

$$2\frac{\epsilon}{\gamma}\pi_{t+s} + \Delta\lambda^\pi_{t+s} - \Delta\lambda^b_{t+s} = 0$$

and taxation,

$$-\frac{\overline{\tau}}{1-\overline{\tau}}\gamma\lambda^\pi_{t+s} - \frac{\overline{w}\overline{N}}{\overline{b}}\frac{\overline{\tau}}{1-\overline{\tau}}\lambda^b_{t+s} = 0$$

These imply that the lagrange multiplier for debt follows a random walk, and that inflation is zero. From the NKPC, this in turn implies the following income tax rule,

$$\varphi y^g_{t+s} + \sigma c^g_{t+s} + \frac{\overline{\tau}}{1-\overline{\tau}}\tau^g_{t+s} = 0$$

The remaining focs are static except for the foc for consumption. However utilising the foc for taxation allows us to write

$$2\sigma\theta c^g_{t+s} - \gamma\sigma\lambda^\pi_{t+s} - \theta\lambda^y_{t+s} - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}\sigma\lambda^b_{t+s} = 0$$

(2) s=0, Commitment Policy

In the initial period, $s = 0$, the initial values of the lagrange multipliers associated with the problem will be zero.Since the only focs which are dynamic are the lagrange multipliers for inflation and the budget constraint, in the initial period inflation will be determined by,

$$2\frac{\epsilon}{\gamma}\pi_{t+s} + \lambda^{\pi}_{t+s} - \widetilde{\lambda}^{b,j}_{t+s} = 0$$

where $\widetilde{\lambda}^{b,j}_{t}$ is the lagrange-multiplier associated with the government's budget constraint under optimal (non-timeless) commitment where $j =$ [real,nom] depending on whether debt is real or nominal. From the foc for taxation this can be rewritten as,

$$2\epsilon\pi_t = (\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma)\widetilde{\lambda}^{b,j}_{t} \tag{15}$$

This captures the extent to which fiscal stress generates inflation in the initial period. (Note that if debt was real rather than nominal there would still be a time inconsistency problem implying inflation in the initial period is still given by this expression, although the size of the lagrangian associated with the budget constraint will be different - see below.) This implies that policy instruments are moved in such a way as to generate inflation. We shall analyse the exact pattern of policy response below.

The tax rule in the initial period is given by,

$$\gamma(\varphi y^g_t + \sigma c^g_t + \frac{\overline{\tau}}{1-\overline{\tau}}\tau^g_t) = \frac{1}{2\epsilon}(\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma)\widetilde{\lambda}^{b,j}_{t}$$

implying that the initial period's inflation rate is given by, $\frac{1}{2\epsilon}(\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma)\widetilde{\lambda}^{b,j}_{t}$.

Similarly the foc for consumption in the initial period is given by,

$$2\sigma\theta c^g_t - \theta\lambda^y_t - \sigma(\frac{\overline{w}\overline{N}(1-\overline{\tau})}{\overline{b}} + \beta)\widetilde{\lambda}^{b,j}_{t} = 0$$

and the remaining focs are, for government spending,

$$2\sigma(1-\theta)g^g_t - (1-\theta)\lambda^y_t + \frac{\overline{G}}{\overline{b}}\widetilde{\lambda}^{b,j}_{t} = 0$$

the output gap,

$$2\varphi y^g_t - \gamma\varphi\lambda^{\pi}_t + \lambda^y_t - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(1+\varphi)\widetilde{\lambda}^{b,j}_{t} = 0$$

Along with the definition of output,

$$y^g_{t+s} = (1-\theta)g^g_{t+s} + \theta c^g_{t+s}$$

these can be solved in terms of the lagrange multiplier associated with debt to give the equations in the main text.

Appendix 3 - Discretionary Policy

(1) Deriving the Bellman equation

The first problem we face is in formulating a recursive problem when our model contains expectations of the future value of variables, in particular consumption, $E_t c^g_{t+1}$ and inflation, $E_t \pi_{t+1}$. However, since we have a linear-quadratic form for our problem we can hypothesize a solution for these endogenous variables of the form,

$$E_{t-1} c^g_t = \mathbf{C}\mathbf{S}_{t-1}$$

$$E_{t-1} \pi_t = \mathbf{F}\mathbf{S}_{t-1} \tag{16}$$

where $\mathbf{C} = \begin{bmatrix} c1 & c2 & c3 & c4 & c5 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} f1 & f2 & f3 & f4 & f5 \end{bmatrix}$ are two $1x5$ vectors of undefined constants and $\mathbf{S}_{t-1} = \begin{bmatrix} \widehat{b}_t \\ a_{t-1} \\ \widehat{\xi}^N_{t-1} \\ \mu_{t-1} \\ \widehat{\xi}_{t-1} \end{bmatrix}$ and $\mathbf{u}_t = \begin{bmatrix} c^g_t \\ \tau^g_t \\ g^g_t \end{bmatrix}$ are the vectors of state and control variables respectively.[12]

Using the former of these we can write the equations describing the evolution of the state variables[13] as,

$$\mathbf{B0S}_t = \mathbf{B1S}_{t-1} + \mathbf{B2u}_t + \mathbf{B3}\boldsymbol{\xi}_t$$

$$\mathbf{B0} = \begin{bmatrix} B0_{1,1} & B0_{1,2} & B0_{1,3} & B0_{1,4} & B0_{1,5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned} \text{where } B0_{1,1} &= \beta - \sigma\beta c1,\ \mathrm{B0}_{1,2} = (\sigma(1-\rho_a\beta) - (\sigma-1)(1-\beta))\frac{(1+\varphi)}{\sigma+\varphi} - \sigma\beta c2 \\ B0_{1,3} &= -(\sigma(1-\rho_N\beta) - (\sigma-1)(1-\beta))\frac{1}{\sigma+\varphi} - \sigma\beta c3 \\ B0_{1,4} &= \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}} - \sigma\beta c4, \text{ and, } B0_{1,5} = \sigma\beta(1-\rho_\xi) - \sigma\beta c5 \end{aligned}$$

[12] We treat the consumption gap as a control variable since the monetary authorities have perfect control of this variable by varying interest rates.

[13] In this section we make the empirically plausible assumption that debt is denominated in nominal terms.

$$
\begin{aligned}
\mathbf{B1} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & \rho_N & 0 & 0 \\ 0 & 0 & 0 & \rho_\mu & 0 \\ 0 & 0 & 0 & 0 & \rho_\xi \end{bmatrix}, \text{ and } \mathbf{B2} = \begin{bmatrix} B2_{1,1} & B2_{1,1} & B2_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\text{where } B2_{1,1} &= \frac{(\varphi\theta+\sigma)(-(1+\gamma)\overline{\frac{B}{Y}} - (1-\theta)) - \theta(1-\theta) - (1-\beta)\theta\overline{\frac{B}{Y}} + \beta\varphi\theta\overline{\frac{B}{Y}}}{\overline{\frac{B}{Y}}}, \\
B2_{1,2} &= \frac{(-(2-\theta) - (1-\beta+\gamma)\overline{\frac{B}{Y}})((1-\beta)\overline{\frac{B}{Y}} + (1-\theta))}{\overline{\frac{B}{Y}}} \text{ and,} \\
B2_{1,3} &= \frac{((1+\varphi)(\theta - (1-\beta)\overline{\frac{B}{Y}}) - \varphi(1+\gamma\overline{\frac{B}{Y}}))(1-\theta)}{\overline{\frac{B}{Y}}}
\end{aligned}
$$

and $\boldsymbol{\xi}_t$ is a vector of iid shocks to our shock processes. This allows us to rewrite the equation of motion for the state variables as,

$$\mathbf{S}_t = \mathbf{D1S}_{t-1} + \mathbf{D2u}_t + \mathbf{D3}\boldsymbol{\xi}_t \tag{17}$$

where

$$\mathbf{D1} = \mathbf{B0}^{-1}\mathbf{B1}$$

and,

$$\mathbf{D2} = \mathbf{B0}^{-1}\mathbf{B2}$$
$$\mathbf{D3} = \mathbf{B0}^{-1}\mathbf{B3}$$

Similarly we can write the evolution of inflation as follows,

$$E_t\boldsymbol{\pi}_{t+1} = \mathbf{A1}\boldsymbol{\pi}_t + \mathbf{A2u}_t \tag{18}$$

where

$$
\begin{aligned}
\mathbf{A1} &= \begin{bmatrix} \frac{1}{\beta} \end{bmatrix} \text{ and,} \\
\mathbf{A2} &= \begin{bmatrix} -\frac{\gamma(\varphi\theta+\sigma)}{\beta} & -\frac{\gamma((1-\beta)\overline{\frac{B}{Y}}+(1-\theta))}{\beta} & -\frac{\gamma\varphi(1-\theta)}{\beta} \end{bmatrix}
\end{aligned}
$$

Leading equation (16) forward one period and utilising the equation describing the evolution of the state variables, we can write,

$$\mathbf{FD1S}_{t-1} + \mathbf{FD2u}_t + \mathbf{FD3}\boldsymbol{\xi}_t = \mathbf{A1}\boldsymbol{\pi}_t + \mathbf{A2u}_t$$

Solving for inflation,

$$\boldsymbol{\pi}_t = \mathbf{C1S}_{t-1} + \mathbf{C2u}_t + \mathbf{C3}\boldsymbol{\xi}_t$$

where

$$\mathbf{C1} \equiv [\mathbf{A1}]^{-1}[\mathbf{FD1}]$$

$$\mathbf{C2} \equiv -[\mathbf{A1}]^{-1}[\mathbf{A2} - \mathbf{FD2}]$$

and,

$$\mathbf{C3} \equiv [\mathbf{A1}]^{-1}[\mathbf{FD3}]$$

These allow us to derive equation (14) in the main text

(2) Solving the Bellman equation

The first-order conditions with respect to the control variables from solving (14) are then given by,

$$2\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t + (\mathbf{Q} + \mathbf{Q}')\mathbf{u}_t + \beta\mathbf{D2}' E_t \frac{\partial V(\mathbf{S}_t; \boldsymbol{\xi}_{t+1})}{\partial \mathbf{S}_t} = 0$$

Note that since $\mathbf{Q}$ has a middle row and column of zeros, the focs will not contain any terms in the tax instrument, such that we effectively have three focs in four unknowns, $\pi_t, y^g_{t,} g^g_t$ and $E_t \frac{\partial V(\widehat{b}_{t+1};\boldsymbol{\xi}_{t+1})}{\partial \widehat{b}_{t+1}}$. These can be arranged as the following linear target criteria,

$$2\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t + \mathbf{H}\begin{bmatrix} c^g_t \\ g^g_t \\ E_t \frac{\partial V(\widehat{b}_{t+1};\boldsymbol{\xi}_{t+1})}{\partial \widehat{b}_{t+1}} \end{bmatrix} = 0$$

where $\mathbf{H} = \begin{bmatrix} 2\mathbf{Q}_{11} & \mathbf{Q}_{1,3} + \mathbf{Q}_{3,1} & \beta\mathbf{D2}_{1,1} \\ \mathbf{Q}_{2,1} + \mathbf{Q}_{1,2} & \mathbf{Q}_{2,3} + \mathbf{Q}_{3,2} & \beta\mathbf{D2}_{2,1} \\ \mathbf{Q}_{1,3} + \mathbf{Q}_{3,1} & 2\mathbf{Q}_{33} & \beta\mathbf{D2}_{3,1} \end{bmatrix}$ and $\mathbf{Q}_{i,j}$ denotes the element contained in row i, column j of matrix $\mathbf{Q}$. This can be solved to yield the following target criteria under discretion,

$$\begin{bmatrix} c^g_t \\ g^g_t \\ E_t \frac{\partial V(\widehat{b}_{t+1};\boldsymbol{\xi}_{t+1})}{\partial \widehat{b}_{t+1}} \end{bmatrix} = -2\mathbf{H}^{-1}\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t \equiv \mathbf{X}\pi_t \tag{19}$$

where

$$\mathbf{X} = \begin{bmatrix} -\frac{(\sigma(\sigma+\varphi-\frac{\sigma+\varphi}{\sigma}\theta(1-\theta))-\frac{\overline{B}}{\overline{Y}}\sigma(\sigma\beta+\theta(1-\beta)+\varphi\beta(1-\theta)))\varepsilon}{(\varphi+\sigma)\sigma\theta((1-\theta)+(1-\beta)\frac{\overline{B}}{\overline{Y}}+1+\frac{\overline{B}}{\overline{Y}}\gamma)} \\ -\frac{(\theta(\varphi+\sigma)+\sigma(\beta(1+\varphi)-1)\frac{\overline{B}}{\overline{Y}})\varepsilon}{\sigma(\varphi+\sigma)((1-\theta)+(1-\beta)\frac{\overline{B}}{\overline{Y}}+1+\frac{\overline{B}}{\overline{Y}}\gamma)} \\ -\frac{2(-\gamma\frac{\overline{B}}{\overline{Y}}(1-\sigma c1-f1)+f1(2-\theta)+f1\frac{\overline{B}}{\overline{Y}}(1-\beta))\varepsilon}{\gamma((1-\theta)+(1-\beta)\frac{\overline{B}}{\overline{Y}}+1+\frac{\overline{B}}{\overline{Y}}\gamma)} \end{bmatrix}$$

Note that the first two elements do not depend upon our 'guess' parameters,$f1..f5$ and $c1..c5$, and imply that there is a linear relationship between the consumption gap and inflation and between the government spending gap and inflation under discretion. The implications of equation (19) for how policy instruments move debt over time are discussed in the main text.

The first order conditions with respect to the state variables are given by,

$$\frac{\partial V(\mathbf{S}_{t-1};\boldsymbol{\xi}_t)}{\partial \mathbf{S}_{t-1}} = 2\mathbf{C1}'\mathbf{R}\boldsymbol{\pi}_t + \beta \mathbf{D1}' E_t \frac{\partial V(\mathbf{S}_t;\boldsymbol{\xi}_{t+1})}{\partial \mathbf{S}_t}$$

Since the state variables relating to the shock process are exogenous, we can focus on the first row of these first-order conditions to obtain,

$$\frac{\partial V(\widehat{b}_t;\boldsymbol{\xi}_t)}{\partial \widehat{b}_t} = 2\mathbf{C1}_{1,1}\mathbf{R}\boldsymbol{\pi}_t + \beta \mathbf{D1}_{1,1} E_t \frac{\partial V(\widehat{b}_{t+1};\boldsymbol{\xi}_{t+1})}{\partial \widehat{b}_{t+1}}$$

Using the focs (i.e applying the envelope theorem),

$$\frac{\partial V(\widehat{b}_t;\boldsymbol{\xi}_t)}{\partial \widehat{b}_t} = \mathbf{W}\pi_t$$

where $\mathbf{W} \equiv [2\mathbf{C1}_{1,1}\mathbf{R} + \beta \mathbf{D1}_{1,1}\mathbf{X}_{3,1}]$.

Leading this one period and applying expectations,

$$\begin{aligned} E_t \frac{\partial V(\widehat{b}_{t+1};\boldsymbol{\xi}_{t+1})}{\partial \widehat{b}_{t+1}} &= \mathbf{W} E_t \boldsymbol{\pi}_{t+1} \\ &= \mathbf{WFS}_t \\ &= \mathbf{WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t + \mathbf{D3}\boldsymbol{\xi}_t] \end{aligned}$$

substituting back into the focs,

$$2\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t + (\mathbf{Q}+\mathbf{Q}')\mathbf{u}_t + \beta \mathbf{D3WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t + \mathbf{D3}\boldsymbol{\xi}_t] = 0$$

where $\mathbf{D3} = \begin{bmatrix} \mathbf{D2}_{1,1} \\ \mathbf{D2}_{1,2} \\ \mathbf{D2}_{1,3} \end{bmatrix}$. Eliminating inflation,

$$2\mathbf{C2}'\mathbf{R}[\mathbf{C1S_{t-1}} + \mathbf{C2u_t} + \mathbf{C3}\boldsymbol{\xi}_\mathbf{t}] + (\mathbf{Q}+\mathbf{Q}')\mathbf{u}_t + \beta \mathbf{D3WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t + \mathbf{D3}\boldsymbol{\xi}_t] = 0$$

and solving for control variables,

$$\mathbf{u_t} = -[\mathbf{U1}]^{-1}\mathbf{U2S}_{t-1} - [\mathbf{U1}]^{-1}2\mathbf{C2}'\mathbf{RC3}\boldsymbol{\xi}_\mathbf{t}$$

where $\mathbf{U1} = [2\mathbf{C2}'\mathbf{RC2} + [\mathbf{Q}+\mathbf{Q}'] + \beta\mathbf{BD3WFD2}]$, $\mathbf{U2} = [2\mathbf{C2}'\mathbf{RC1} + \beta\mathbf{D3WFD1}]$ and $\mathbf{U3} = [2\mathbf{C2}'\mathbf{RC3} + \beta\mathbf{D3WFD3}]$. The solution for inflation is now given as,

$$\boldsymbol{\pi}_t = [\mathbf{C1} - \mathbf{C2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2}]\mathbf{S}_{t-1} + [\mathbf{C3} - \mathbf{C2}[\mathbf{U1}]^{-1}\mathbf{U3}]\boldsymbol{\xi}_\mathbf{t}$$

Taking expectations at time t-1,

$$E_{t-1}\boldsymbol{\pi}_t = [\mathbf{C1} - \mathbf{C2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2}]\mathbf{S}_{t-1}$$

However, this solution is a function of the undetermined coefficients, $f1..$ $f5$ and $c1..c5$, which can be derived by equating coefficients,

$$\begin{aligned} \mathbf{F} &= \mathbf{C1} - \mathbf{C2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2} \\ \mathbf{C} &= \mathbf{X}_{1,1}\mathbf{F} \end{aligned}$$

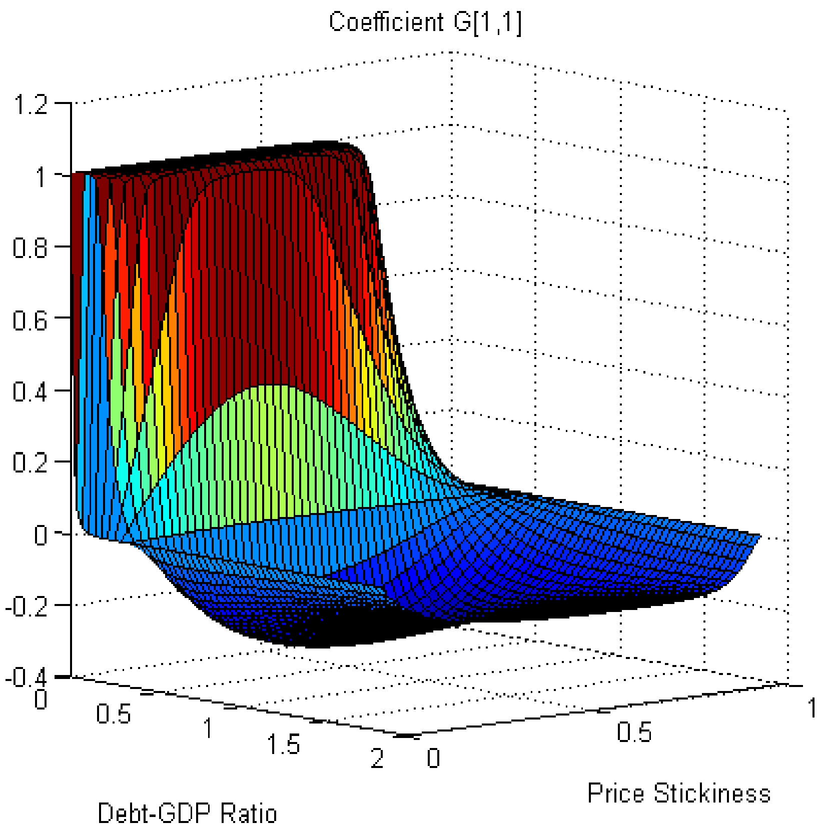

Figure 1: Coefficient G[1,1] as a function of the debt/gdp ratio and degree of price stickiness.

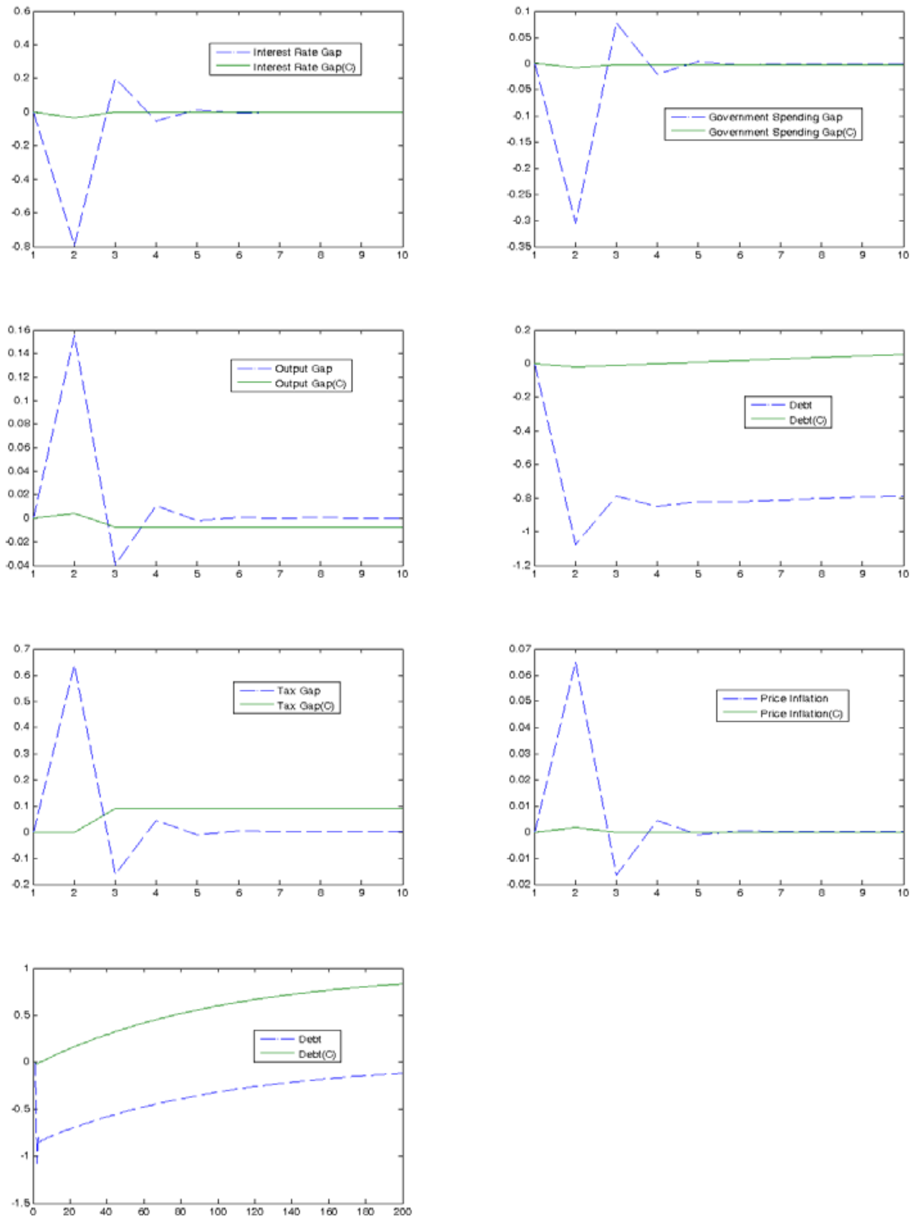

Figure 2: Response to a 1% Technology Shock Under Commitment and Discretion (Notes to Figure: Time period in graph in bottom left is 200 periods rather than 10.)

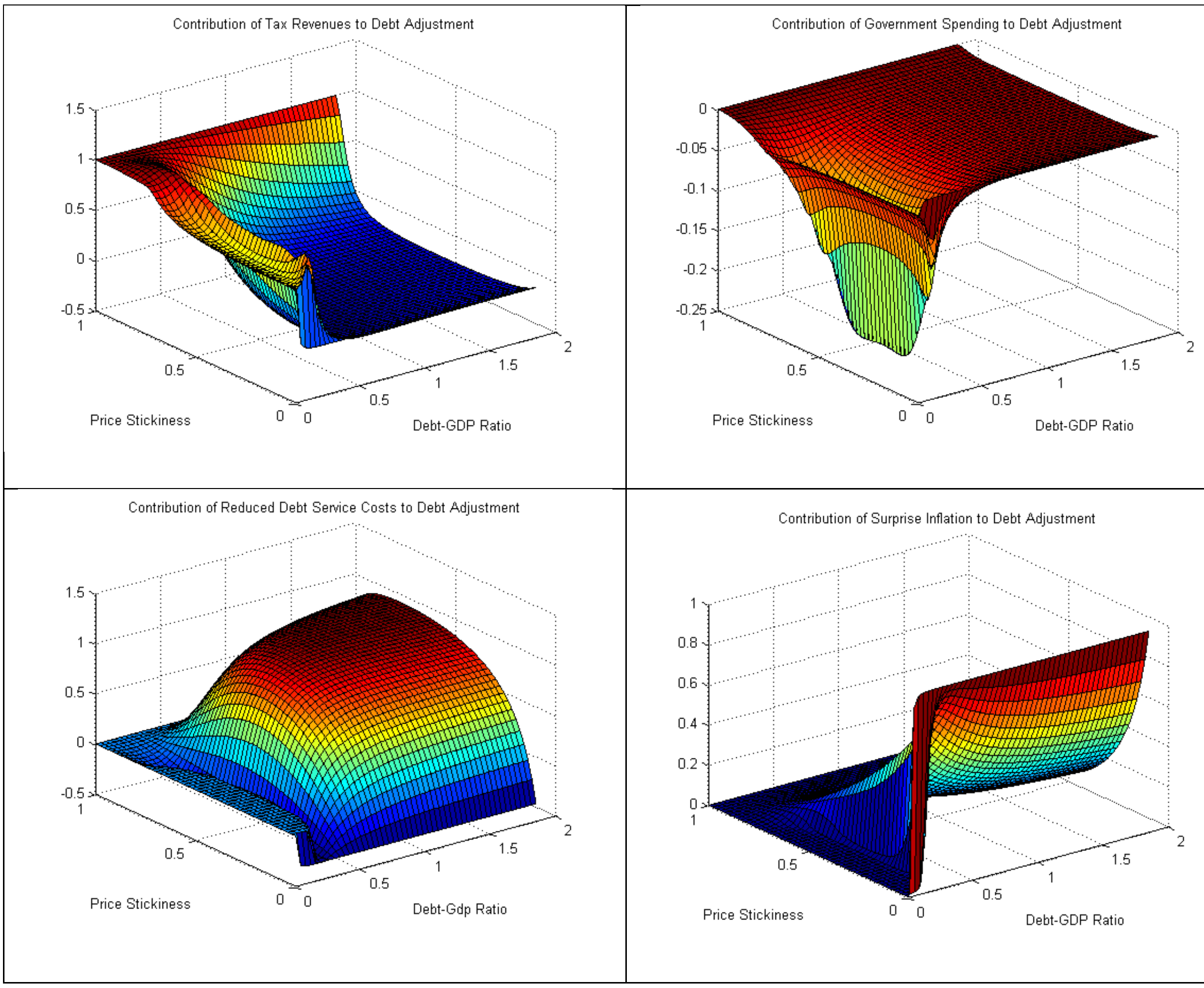

Figure 3: The contribution of alternative policy instruments to debt adjustment under discretion.